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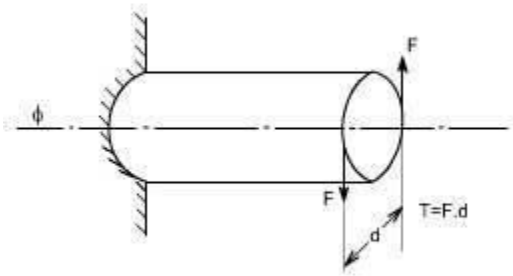
Chapter 1

Members Subjected to Torsional Loads

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross 1 section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

GENERATION OF SHEAR STRESSES

The physical understanding of the phenomena of setting up of shear stresses in a shaft subjected to a torsion may be understood from the figure 1-3.

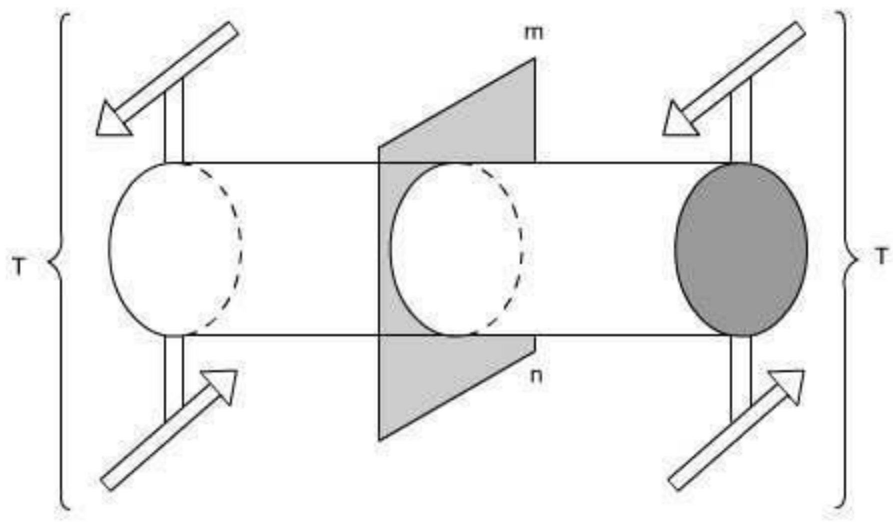


Fig 1: Here the cylindrical member or a shaft is in static equilibrium where T is the resultant external torque acting on the member. Let the member be imagined to be cut by some imaginary plane 1mn'.

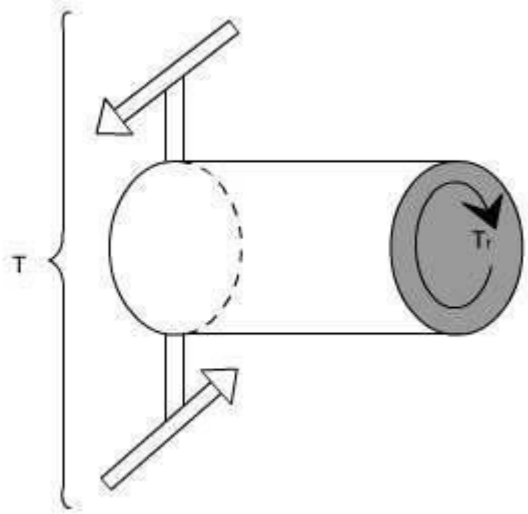


Fig 2: When the plane 1mn' cuts remove the portion on R.H.S. and we get a fig 2. Now since the entire member is in equilibrium, therefore, each portion must be in equilibrium. Thus, the member is in equilibrium under the action of resultant external torque T and developed resisting Torque T_r .

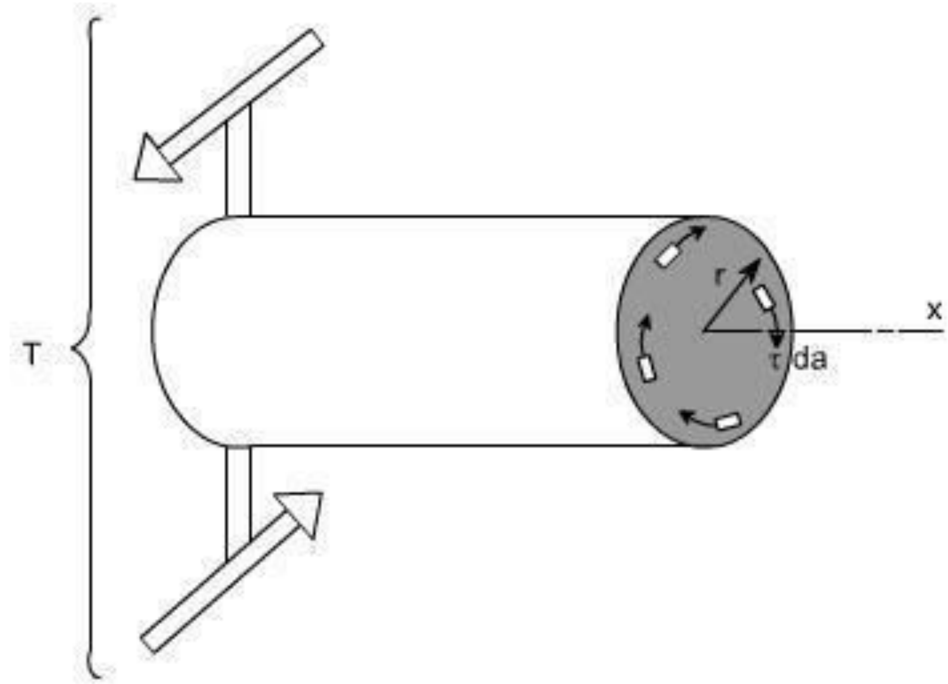


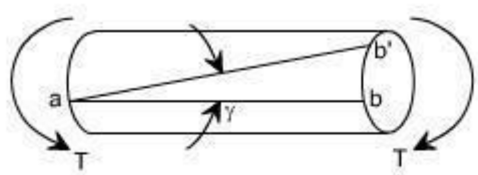
Fig 3: The Figure shows that how the resisting torque T_r is developed. The resisting torque T_r is produced by virtue of an infinites mal shear fo rces acting on the plane perpendicular to the axis of the shaft. Obviously such shear forces would be develo ped by virtue of sheer stresses.

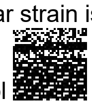
Therefore we can say that when a particular member (say shaft in this case) is subjected to a torque, the result would be that on any element there will be shear stresses acting. While on other faces the complementary sheer forces come into picture. Thus, we can say that when a member is sub jected to torque, an element of this member will be subjected to a state of pure shear.

Shaft: The shafts are the machine elements which are used to transmit power in machines.

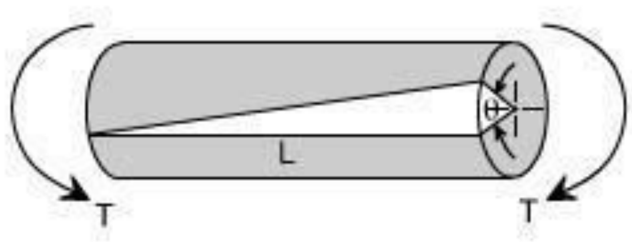
Twisting Moment: The twisting mo ment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under considera tion. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a 1 b is marked on the surface of the unloaded bar, then afte r the twisting moment 'T' has been applied this li ne moves to ab'. The angle 1γ' measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.



Modulus of Elasticity in shear: T he ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol 

Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its l ngth, than the angle θ through which one end of t he bar will twist relative to the other is known is the angle of twist.

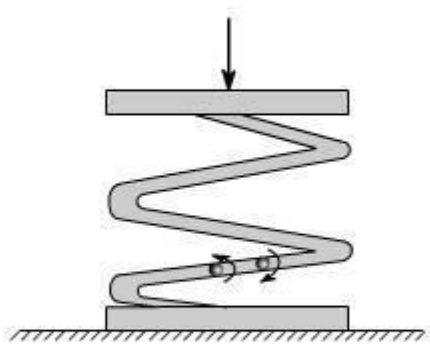


- Despite the difference s in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.

  In torsion the members are subjected to moments (couples) in planes normal to their axes.

- For the purpose of desiging a circular shaft to withstand a given torqu e, we must develop an equation giving the relation between twisting moment, ma ximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of veh icle suspension system employ torsion al springs. Indeed, even coil springs are really curved mem bers in torsion as shown in figure.



- Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers.

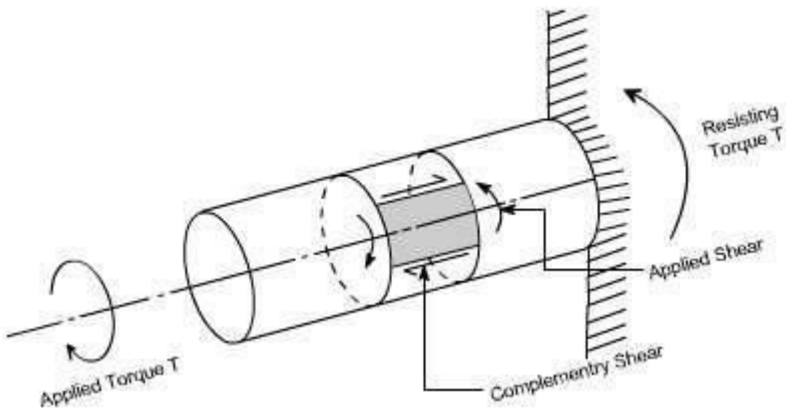
Simple Torsion Theory or Develo pment of Torsion Formula : Here we are basically inter ested to derive an equation between the relevant parameters

Relationship in Torsion: 

1 st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area.

2 nd Term: This refers to stress, and the stress increases as the distance from the axis increases.

3 rd Term: it refers to the deformation and contains the terms modulus of rigidity & combined term (θ / l) which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross 1 sectional area of the shaft.



Refer to the figure shown above where a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear, the moment of resistance developed by the shear stresses being every where equal to the magnitude, and opposite in sense, to the applied torque. For the purpose of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary make the following base assumptions.

Assumption:

(i) The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

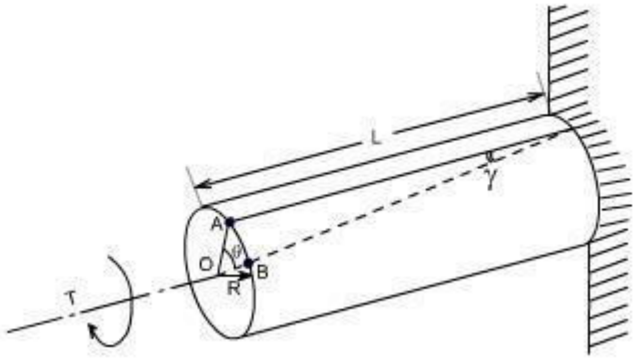
(ii) The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle θ , point A moves to B, and AB subtends an angle 1 γ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = Rθ

= L γ [since L and γ also constitute the arc AB]

Thus, γ = Rθ / L (1)

From the definition of Modulus of ri gidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

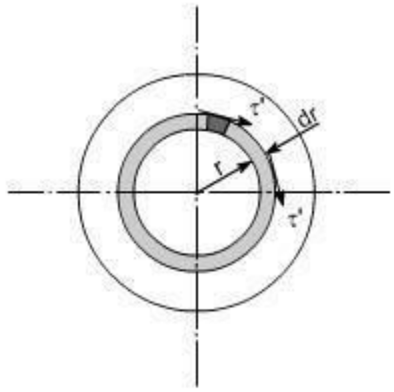
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress_ '.



The force set up on each element

= stress x area

= τ' x 2π r dr (approximately)

This force will produce a moment or torque about the center axis of the shaft.

= $\tau' . 2 \pi r dr . r$

= $2 \pi \tau' . r^2 . dr$

The total torque T on the section, will be the sum of all the contributions.

Since τ' is a function of r, because it varies with radius so writing down_ ' in terms of r from th e equation (1).

i.e $\tau' = \frac{G\theta.r}{L}$

we get $T = \int_0^R 2\pi \frac{G\theta}{L}.r^3 dr$

$$T = \frac{2\pi G\theta}{L}\int_0^R r^3 dr$$

$$= \frac{2\pi G\theta}{L}.\left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{G\theta}{L}.\frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L}.\frac{\pi R^4}{2}$$

$$= \frac{G\theta}{L}.\left[\frac{\pi d^4}{32}\right]$$ now substituting R = d/2

$$= \frac{G\theta}{L}.J$$

since $\frac{\pi d^4}{32}$ = J the polar moment of inertia

or $\frac{T}{J} = \frac{G\theta}{L}$ ......(2)

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

[D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

θ = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist

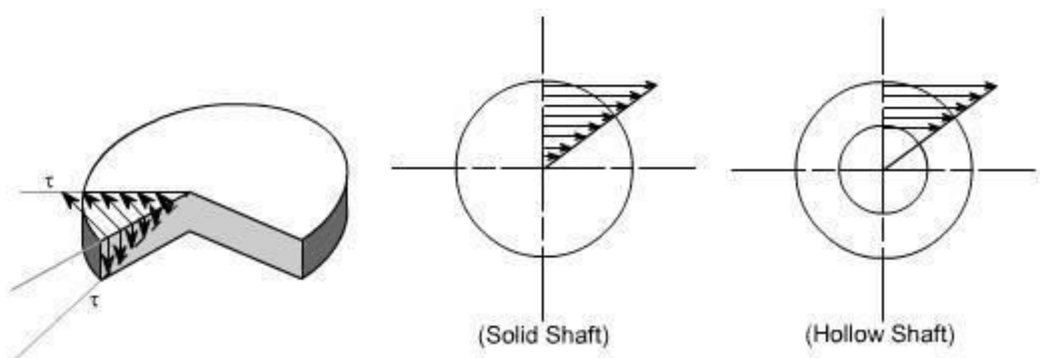
i.e, k = T /_ _= GJ / L

Power Transmitted by a shaft : If T is the applied Torque and ω is the angular velocity of the shaft, then the power transmitted by the shaft is

Distribution of shear stresses in circular Shafts subjected to torsion :

The simple torsion equation is writt en as

This states that the shearing stress varies directly as the distance 1r' from the axis of the sha ft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress o ccurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} \cdot d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following c onclusion can be drawn

(i) $\tau_{max}^{m} \propto T$

(ii) $\tau_{max}^{m} \propto 1/d^{3}$

Power Transmitted by a shaft:

In practical application, the diamete r of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be tran smitted, speed in rpm 1N' Torque T, the formula connecting

These quantities can be derived as follows

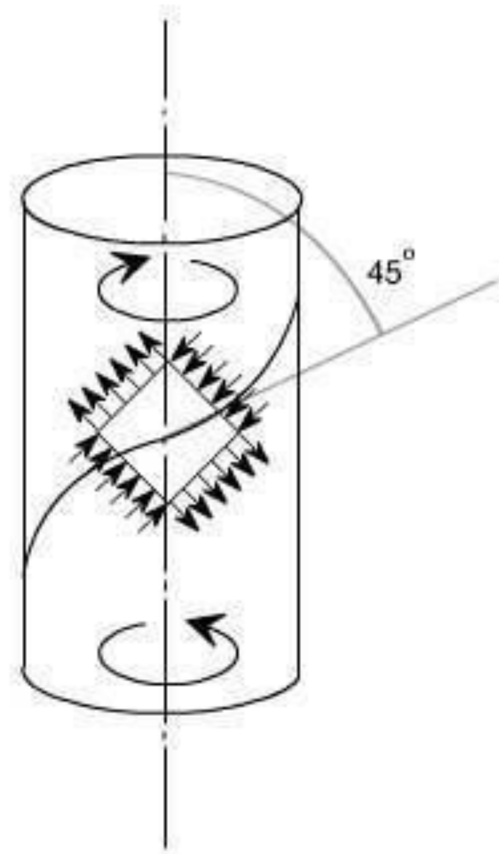
Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

For a ductile material, the plastic flo w begins first in the outer surface. For a material which is weaker in shear longitudinally than transverse ly 1 for instance a wooden shaft, with the fibres parallel t o axis the first cracks will be produced by the she aring stresses acting in the axial section and they will uppe r on the surface of the shaft in the longitudin al direction.

In the case of a material which is weaker in tension than in shear. For instance a, circular sha ft of cast iron or a cylindrical piece of chalk a cracck along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state o f stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses, the ten sile stresses shown will produce a helical crack mentioned .



TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x 1 section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

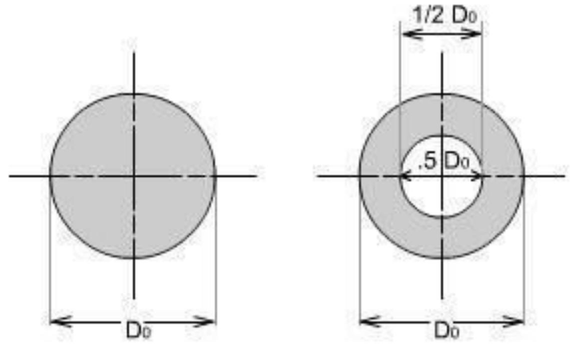
$$= \frac{16T.D_0}{\pi D_0^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the τ_{max^m} in the case of hollow shaft is 6.6% larger then in the case of a so lid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density 'ρ' with $d_i = 1/2\ D_o$



Weight of hollow shaft

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi D_0^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^2}{4}\left[1 - 1/4\right] l \times \rho$$

$$= 0.75 \frac{\pi D_0^2}{4} l \times \rho$$

$$\text{Weight of solid shaft} = \frac{\pi D_0^2}{4} . l . \rho$$

$$\text{Reduction in weight} = (1 - 0.75)\frac{\pi D_0^2}{4} l \times \rho$$

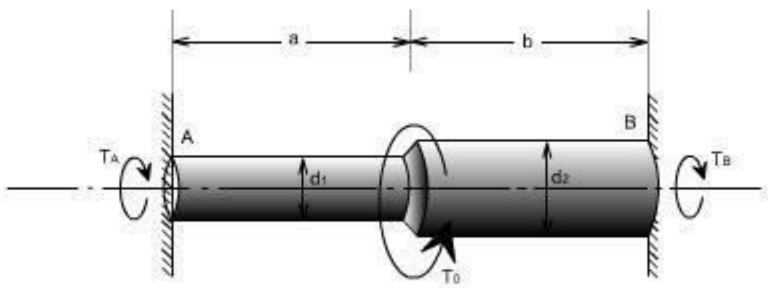
$$= 0.25 \frac{\pi D_0^2}{4} l \times \rho$$

Hence the reduction in weight would be just 25%.

Illustrative Examples :

Problem 1

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. De termine the angle of rotation θ_0 of the shoulder section wh ere T_0 is applied ?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built 1 in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$------ (1)

[from static principles]

Where T_A ,T_B are the reactive torque at the built in ends A and B. wheeras T_0 is the applied t orque

From consideration of consistent d eformation, we see that the angle of twist in each portion o f the shaft must be same.

i.e $\theta_a = \theta_b = \theta_0$

using the relation for angle of twist

$$\frac{T}{J} = \frac{G.\theta}{l}$$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

$$\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B} . \frac{b}{a} \quad (2)$$

N.B: Assuming modulus of rigidity G to be same for the two portions

So the defines the ratio of T_A and T_B

So by solving (1) & (2) we get

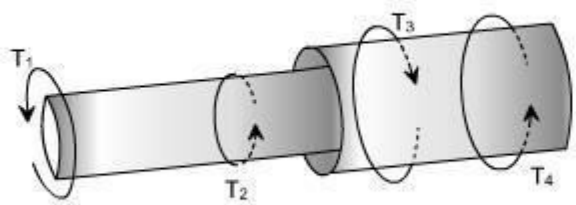
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junctior

$$\theta_0 = \frac{T_0 . a . b}{[J_A . b + J_B . a] G}$$

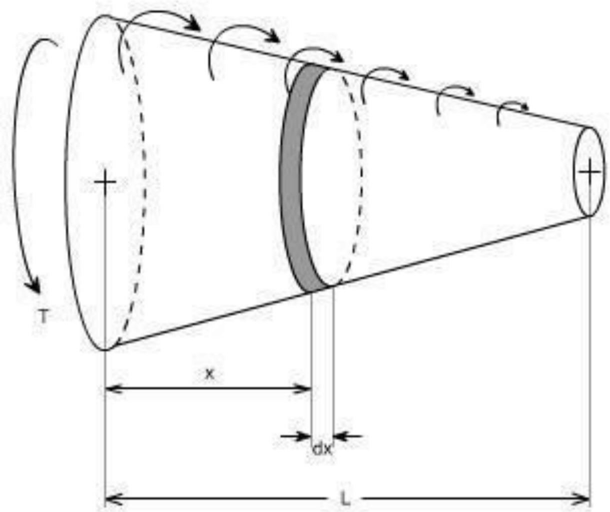
Non Uniform Torsion: The pure to rsion refers to a torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the b ar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes i n cross section is in pure torsion, hence the formul a's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

If either the torque or the cross section changes continuously along the axis of the bar, then the Σ (summation can be replaced by an integral sign (∫). i.e We will have to consider a diffe rential element.



After considering the differential ele ment, we can write

Substituting the expressions for T_x and J_x at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the v alue of angle of twist may be determined.

Closed Coiled helical springs su bjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

or

Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

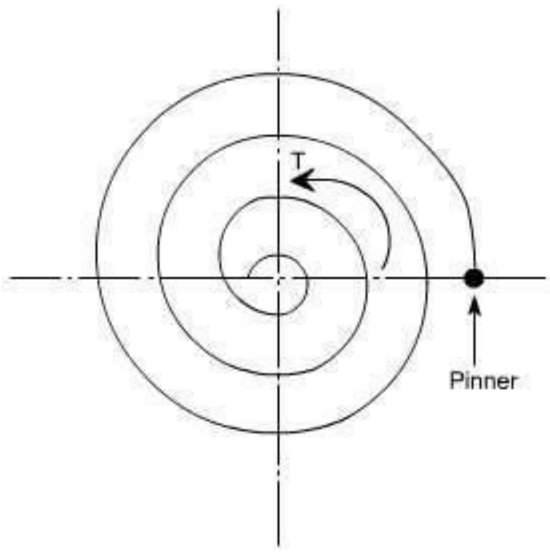
Important types of springs are:

There are various types of springs such as

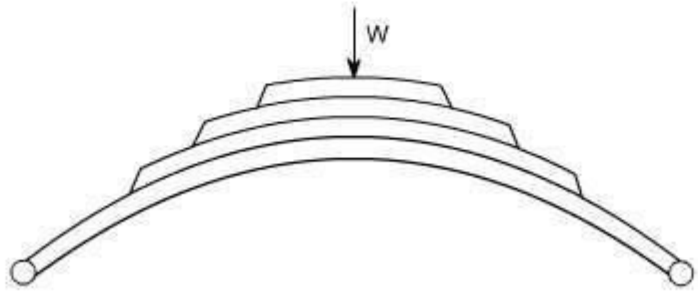
(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.

(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compression due to bending.



(iv) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency . Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.



These type of springs are used in the automobile suspension system.

Uses of springs :

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

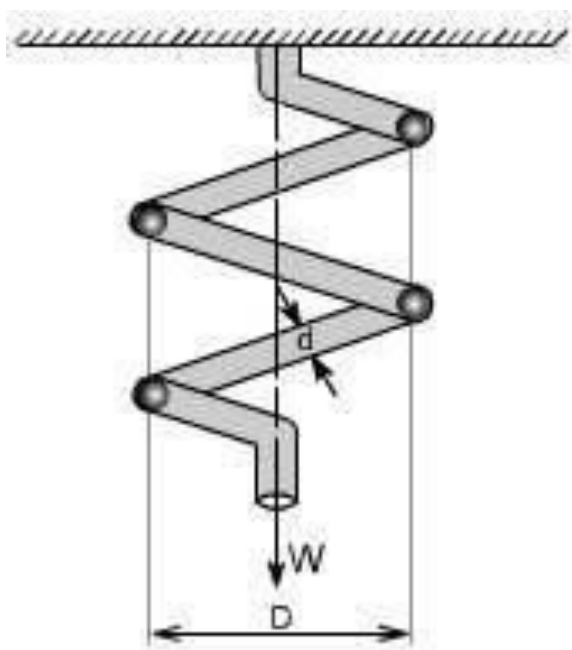
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



Let

W = axial load

D = mean coil diameter

d = diameter of spring wire

n = number of active coils

C = spring index = D / d For circular wires

l = length of spring wire

G = modulus of rigidity

x = deflection of spring

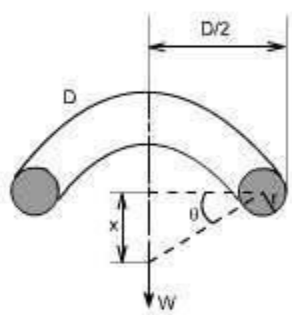
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

$x = D / 2 . \theta$

again $l = \pi$ D n [consider ,one half turn of a close coiled helical spring]



Assumptions: (1) The Bending & shear effects may be neglected

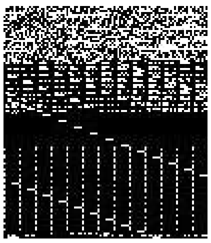
(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a such a spring will be assumed to lie in a plane which is nearly $\perp^r$ to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X 1 section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible

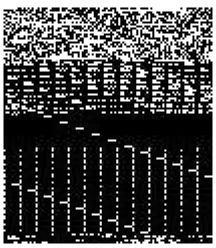
so applying the torsion formula.

Using the torsion formula i.e

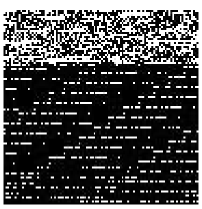
SPRING DEFLECTION



Spring striffness: The stiffness is defined as the load per unit deflection therefore



Shear stress



WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as 

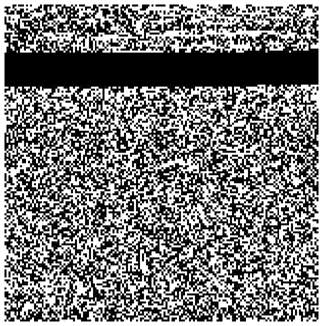
Where C = spring index

= D/d

if we take int account thWahl's f actor than the formula for the shear stres becomes

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion



Example: A close coiled helical spr ing is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm^2 .if the number of active turns or active coils is 8.Esti mate the following:

(i) wire diameter

(ii) mean coil diameter

(iii) weight of the spring.

Assume G = 83,000 N/mm^2 ; ρ = 7 700 kg/m^3

solution :

(i) for wire diametre if W is the axial load, then

Futher, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314d^3)^3.8}{83,000.d^4}$$

$$d = 13.32mm$$

Therefore,

$$D = .0314 \times (13.317)^{3} \text{ mm}$$

=74.15mm

D = 74.15 mm

Weight

mass or weight = volume. density

= area.length of the spring.density of spring materia

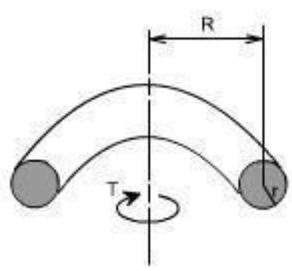
$$= \frac{\pi d^2}{4} . \pi D n . \rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

= 2.0kg

Close coiled helical spring sub jected to axial torque T or axial couple.



In this case the material of the sprin g is subjected to pure bending which tends to reduce Ra ius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximu m bending stress may thus be determined from the bending theory. 

Deflection or wind up angle:

Under the action of an axial torque the deflection of the spring becomes the 1wind 1 up1 angle of the spring which is the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, acc ording to area 1 moment theorem

$$\theta = \int_0^L \frac{M\,dL}{EI} \quad \text{but } M = T$$

$$= \int_0^L \frac{T.dL}{EI} = \frac{T}{EI}\int_0^L dL$$

Thus, as 'T' remains constant

$$\theta = \frac{T.L}{EI}$$

Futher

$$L = \pi D.n$$

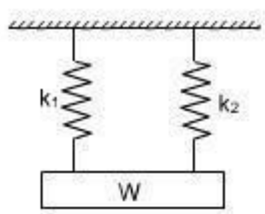
$$I = \frac{\pi d^4}{64}$$

Therefore, on substitution, the value of θ obtained is

$$\boxed{\theta = \frac{64T.D.n}{E.d^4}}$$

Springs in Series: If two springs of different stiffness are joined end on and carry a common load W, they are said to be connected in series a nd the combined stiffness and deflection are given by the following equation.

Springs in parallel: If the two sprin g are joined in such a way that they have a common defl ection 1x' ; then they are said to be connected in parallel.In this care the load carried is shared between the t wo springs and total load W = $W_1 + W_2$



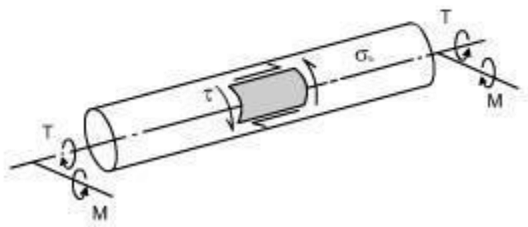
Members Subjected to Combined Loads

Combined Bending & Twisting : In some applications the shaft are simultaneously subject d to bending moment M and Torque T.The Bending moment comes on the shaft due to gravity or Inertia lo ads. So the stresses are set up due to bending moment and Torque.

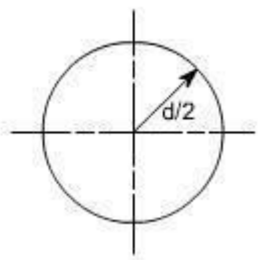
For design purposes it is necessary to find the principal stresses, maximum shear stress, whichever is used as a criterion of failure.

From the simple bending theory equation

If σ_b is the maximum bending stres ses due to bending.



For the case of circular shafts y_{max} m 1 equal to d/2 since y is the distance from the neutral axis.



I is the moment of inertia for circular shafts

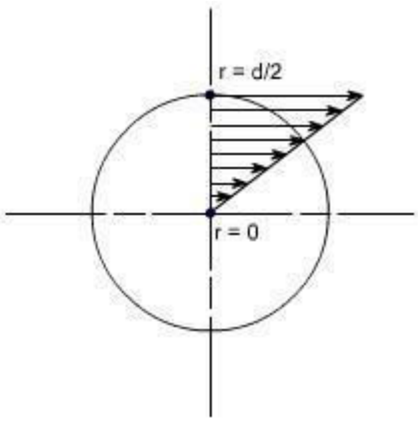
$I = \_ d^4 /64$

Hence then, the maximum bending stresses developed due to the application of bending mo ment M is

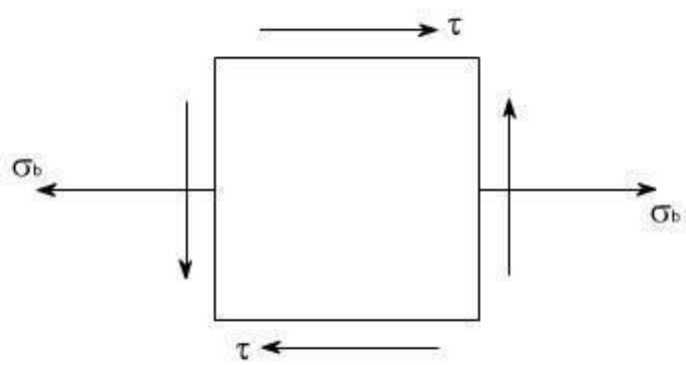
From the torsion theory, the maximum shear stress on the surface of the shaft is given by the torsion equation

Where τ' is the shear stress at any radius r but when the maximum value is desired the value of r should be maximum and the value of r is maximum at r = d/2

The nature of the shear stress distribution is shown below :



This can now be treated as the two 1 dimensional stress system in which the loading in a ve rtical plane in zero i.e. $\sigma_y = 0$ and $\sigma_x = \sigma_b$ and is shown below :



Thus, the principle stresses may be obtained as

$$\sigma_1, \sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^{\ 2}}$$

or

$$\sigma_1 = \frac{\sigma_b}{2} + \frac{1}{2}\sqrt{\sigma_b^{\ 2} + 4\tau_{max}^{\ 2}}$$

$$= \frac{32M}{\pi d^3 .2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3 2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{2.16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Equivalent Bending Moment :

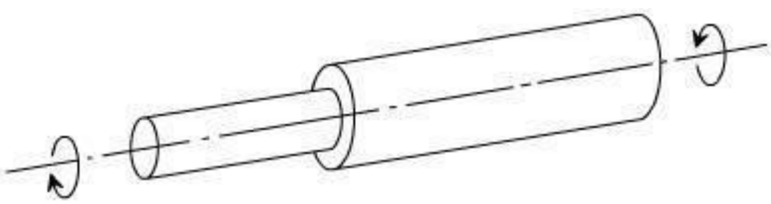
Now let us define the term the equivalent bending moment which acting alone, will produce t he same maximum principal stress or bendin g stress.Let M_e be the equivalent bending moment, then due to bending

where [redacted] is defined as the equivalent torque, which acting alone would produce the same maximum shear stress as produced by the pure torsion

Thus, [illegible]

Composite shafts: (in series)

If two or more shaft of different material, diameter or basic forms are connected together in s uch a way that each carries the same torque, then the shafts are said to be connected in series & the compo site shaft so produced is therefore termed as series 1 connected.



Here in this case the equilibrium of the shaft requires that the torque 1T' be the same throug h out both the parts.

In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion 1 theory to eac h in turn. The composite shaft will therefore be as weak as its weakest component. If relative dimensions o f the various parts are required then a solution is usually effected by equating the torque in each shaft e .g. for two shafts in series

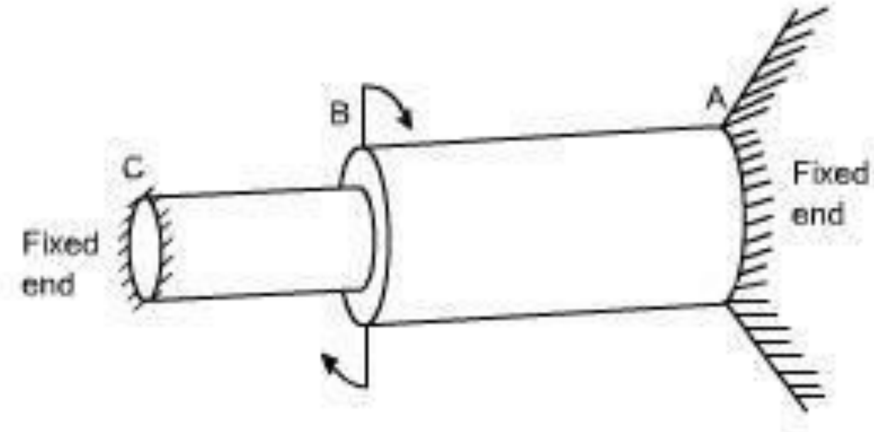
[redacted]

In some applications it is convenien t to ensure that the angle of twist in each shaft are equal i.e. $\theta_1 = \theta_2$, so

that for similar materials in each shaft [redacted]

The total angle of twist at the free end must be the sum of angles $\theta_1 = \theta_2$ over each x - sectio n

Composite shaft parallel connec tion: If two or more shafts are rigidly fixed together such t hat the applied torque is shared between them the n the composite shaft so formed is sad to be connected in parallel.



For parallel connection.

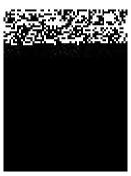
Total Torque $T = T_1 + T_2$

In this case the angle of twist for ea ch portion are equal and 

for equal lengths(as is normaly the case for parallel shafts) 

This type of configuration is statically indeterminate, because we do not know how the applied torque is apportioned to each segment, To deal such type of problem the procedure is exactly the same as we have discussed earlier,

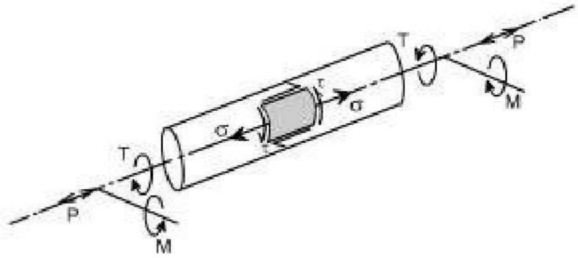
Thus two equations are obtained in terms of the torques in each part of the composite shaft a nd the maximun shear stress in each part can then be found from the relations.

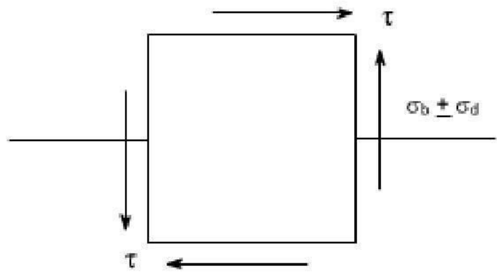


Combined bending, Torsion and Axial thrust:

Sometimes, a shaft may be subjected to a combined bending, torsion and axial thrust. This ty pe of situation arises in turbine propeller shaft

If P = Thrust load





Then σ_d = P / A (stress due to thrust)

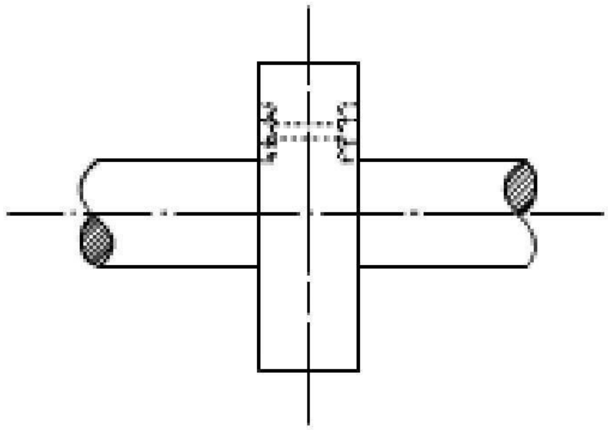
where σ_d is the direct stress depending on the whether the steam is tensile on the whether the stress is tensile or compressive

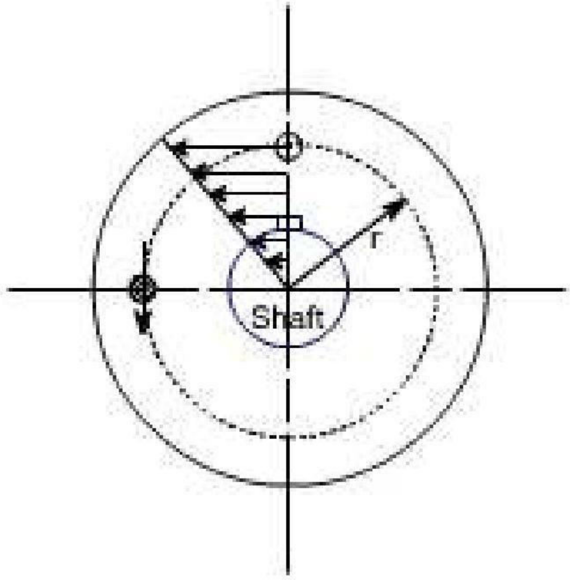
This type of problem may be analyzed as discussed in earlier case.

Shaft couplings: In shaft couplings, the bolts fail in shear. In this case the torque capacity of the coupling may be determined in the following manner

Assumptions:

The shearing stress in any bolt is assumed to be uniform and is governed by the distance from its center to the centre of coupling.





Thus, the torque capacity of the co upling is given as [illegible]

where

d_b = diameter of bolt

τ'_b = maximum shear stress in bolt

n = no. of bolts

r = distance from center of bolt to center of coupling

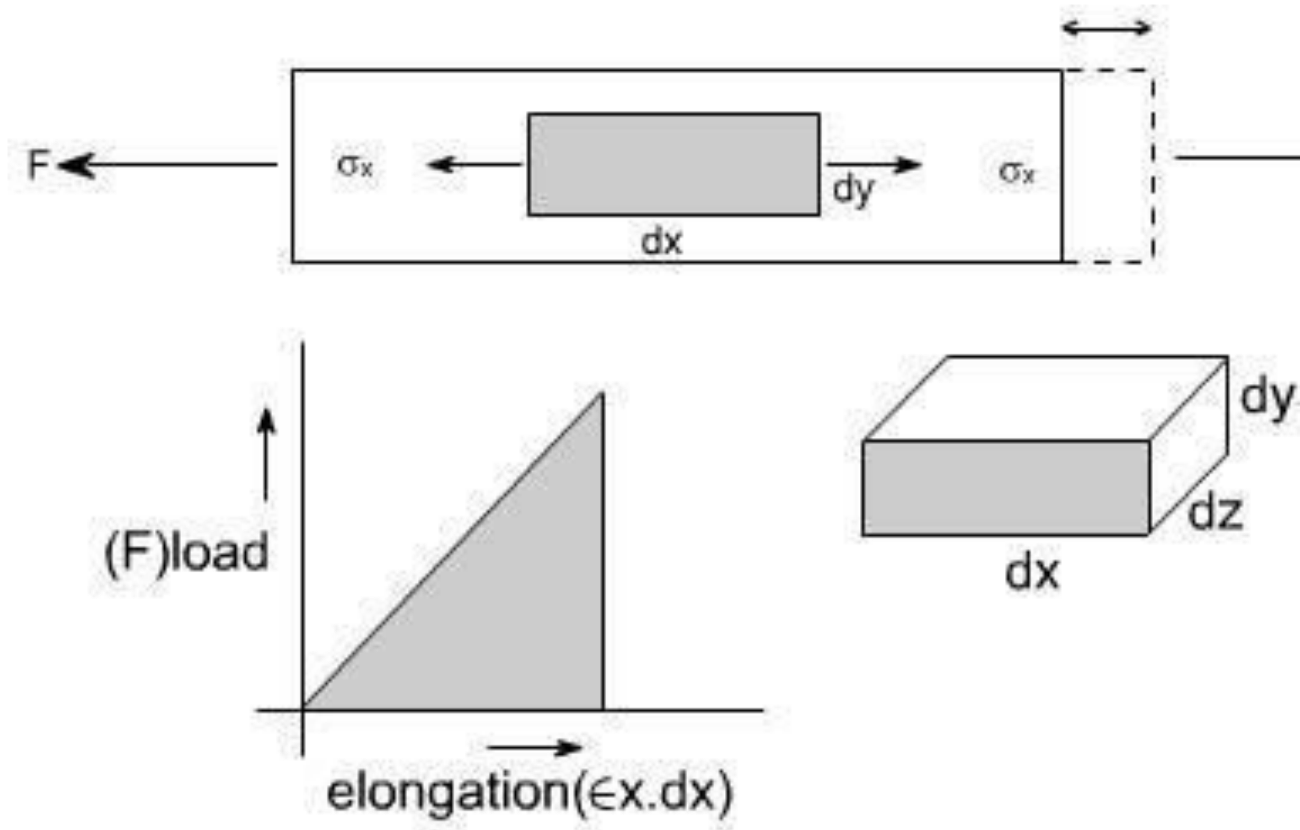
THEORIES OF ELASTIC FAILUR E

While dealing with the design of structures or machine elements or any component of a particular machine the physical properties or chief characteristics of the constituent materials are usually found from the results of laboratory experimen ts in which the components are subject to the simple stress conditions. The most usual test is a simple ten sile test in which the value of stress at yield or fracture is e asily determined.

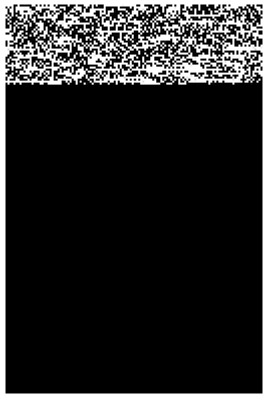
However, a machine part is generally subjected simultaneously to several different types of stresses whose actions are combined therefore, it is necessary to have some basis for determining the allowable working stresses so that failure may not occur. Thus, the function of the theories of elastic failure is to predict from the behavior of materials in a simple tensile test when elastic failure will occur under any conditions of applied stress.

A number of theories have been proposed for the brittle and ductile materials.

Strain Energy: The concept of strain energy is of fundamental importance in applied mechanics. The application of the load produces strain in the bar. The effect of these strains is to increase the energy level of the bar itself. Hence a new quantity called strain energy is defined as the energy absorbed by the bar during the loading process. This strain energy is defined as the work done by load provided no energy is added or subtracted in the form of heat. Some times strain energy is referred to as internal work to distinguish it from external work 1W'. Consider a simple bar which is subjected to tensile force F, having a small element of dimensions dx, dy and dz.



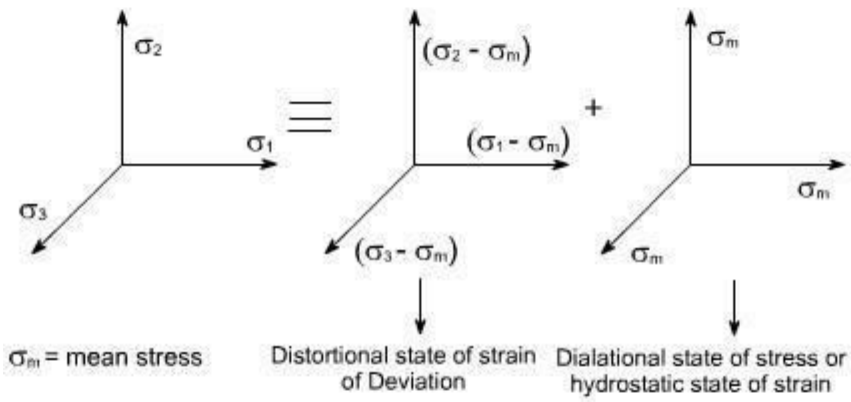
The strain energy U is the area covered under the triangle



A three dimension state of stress re spresented by σ_1, σ_2 and σ_3 may be throught of consistin g of two distinct state of stresses i.e Distortional state of stress

Deviatoric state of stress and dilatio nal state of stress

Hydrostatic state of stresses.



$$\sigma_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$$

Thus, The energy which is stored w ithin a material when the material is deformed is termed a s a strain energy. The total strain energy U_r

$$U_T = U_d + U_H$$

U_d is the strain energy due to the Deviatoric state of stress and U_H is the strain energy due to the Hydrostatic state of stress. Futher, it may be no ted that the hydrostatic state of stress results in change of volume whereas the deviatoric state of stress results in change of shape.

<u>Different Theories of Failure :</u> Th ese are five different theories of failures which are generally used

(iii) Maximum Principal stress theo ry (due to Rankine)

(iv) Maximum shear stress theory (Guest - Tresca)

(v) Maximum Principal strain (Saint - venant) Theory

- Total strain energy per unit volu me (Haigh) Theory
- Shear strain energy per unit volume Theory (Von 1 Mises & Hencky)

In all these theories we shall assum e.

σ_{Yp} = stress at the yield point in the simple tensile test.

$\sigma_{1,\_ 2,\_ 3}$ — the three principal stre sses in the three dimensional complex state of stress systems in order of magnitude.

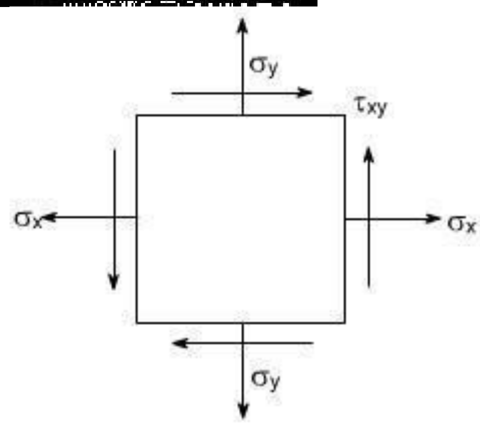
(a) Maximum Principal stress theory :

This theory assume that when the maximum principal stress in a complex stress system reaches the elastic limit stress in a simple tension, failu re will occur.

Therefore the criterion for failure w ould be

$\sigma_1 = \sigma_{yp}$

For a two dimensional complex stre ss system σ_1 is expressed as

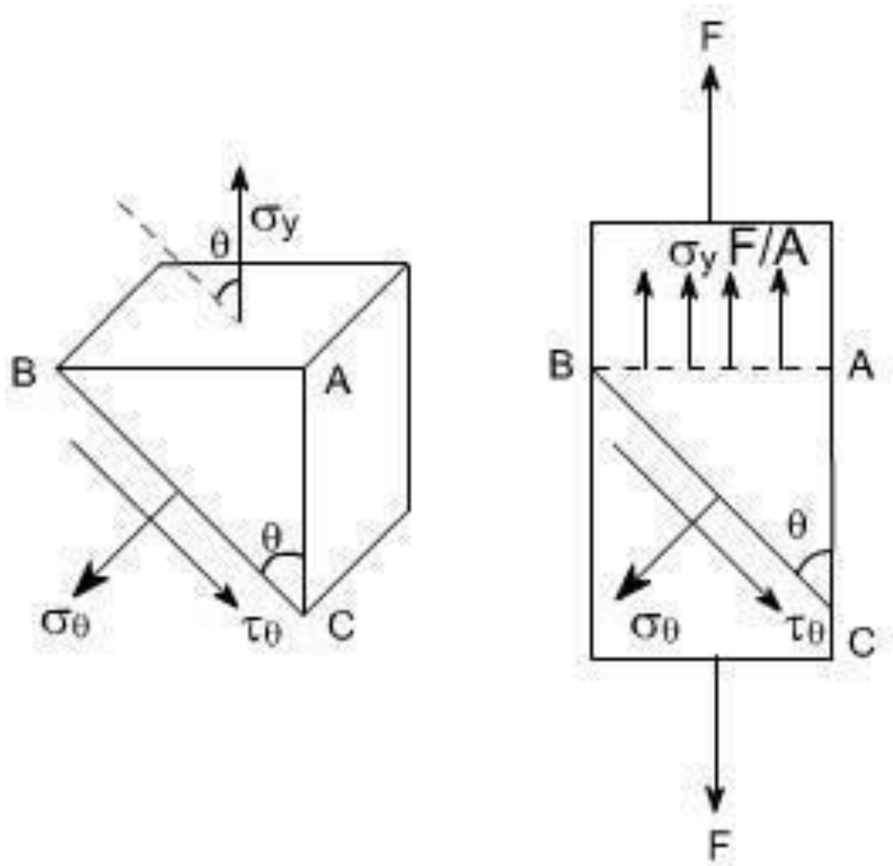


Where σ_x, σ_y and τ_{xy} are the stress es in the any given complex stress system.

(b) Maximum shear stress theory:

This theory states that teh failure c an be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.

The criterion for the failure may be established as given below :



For a simple tension case

$\sigma_\theta = \sigma_y \sin^2\theta$

$\tau_\theta = \frac{1}{2}\sigma_y \sin 2\theta$

$\tau_\theta\big|_{max^m} = \frac{1}{2}\sigma_y$ or

$\tau_{max^m} = \frac{1}{2}\sigma_{yp}$

whereas for the two dimentional complex stress system

$\tau_{max^m} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$

where σ_1 = maximum principle stress

σ_2 = minimum principal stress

so $\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy}$

$\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sigma_{yp} \Rightarrow \sigma_1 - \sigma_2 = \sigma_y$

$\Rightarrow \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy} = \sigma_{yp}$

becomes the criterion for the failure.

(c) Maximum Principal strain the ory :

This Theory assumes that failure o ccurs when the maximum strain for a complex state of stre ss system becomes equals to the strain at yield point in the tensile test for the three dimensional compl ex state of stress system.

For a 3 - dimensional state of stres s system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

$$U_t = 1/2\sigma_1 \epsilon_1 + 1/2\sigma_2 \epsilon_2 + 1/2\sigma_3 \epsilon_3$$

substituting the values of ϵ_1, ϵ_2 and ϵ_3

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \gamma(\sigma_2 + \sigma_3)]$$

$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \gamma(\sigma_1 + \sigma_3)]$$

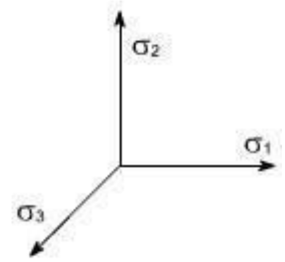
$$\epsilon_3 = \frac{1}{E}[\sigma_3 - \gamma(\sigma_1 + \sigma_2)]$$

Thus, the failure criterion becomes

$$\left(\frac{\sigma_1}{E} - \gamma\frac{\sigma_2}{E} - \gamma\frac{\sigma_3}{E}\right) = \frac{\sigma_{yp}}{E}$$

or

$$\boxed{\sigma_1 - \gamma\sigma_2 - \gamma\sigma_3 = \sigma_{yp}}$$



(d) Total strain energy per unit v olume theory :

The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point a tensile test.

Therefore, the failure criterion becomes

It may be noted that this theory giv es fair by good results for ductile materials.

(e) Maximum shear strain energy per unit volume theory :

This theory states that the failure o ccurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

Hence the criterion for the failure becomes

As we know that a general state of stress can be broken into two components i.e,

- Hydrostatic state of stress (the strain energy associated with the hydrostatic state of stress is known as the volumetric strain energy)

- Distortional or Deviatoric state of stress (The strain energy due to this is known as the sh ear strain energy)

As we know that the strain energy due to distortion is given as

This is the distortion strain energy for a complex state of stress, this is to be equaled to the maximum distortion energy in the simple tension test. In order to get we may assume that one of the principal stress say (_ $_1$) reaches the yield point (_ σ_{yp}) of the material. Thus, putting in above equation $\sigma_2 = \sigma_3 = 0$ we get distortion energy for the simple test i.e

Chapter 2

COLUMNS AND STRUTS

Elastic Stability Of Columns

Introduction:

Structural members which carry co mpressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The bucckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied ex actly along the axis of the Strut.

(c). one part of the material may yie ld in compression more readily than others owing to som e lack of uniformity in the material properties through out the strut.

In all the problems considered so fa r we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or sh orter but remains straight. Under some circumstances however, our assumptions of progressive and sim ple deformation may no longer hold good and the memb er become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the bucklin g load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine w ave provided that the amplitude of wave is kept small.

Theoretically, it is possible for strut s to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditi ons. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stat ed earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the m ember can withstand is therefore related to k. There will b e two principal moments of inertia, if the least of these is taken then the ratio

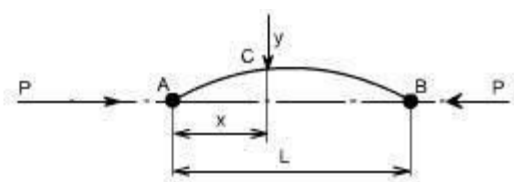
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

Euler's Theory : The struts which fail by buckling can be analyzed by Euler's theory. In the f ollowing sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

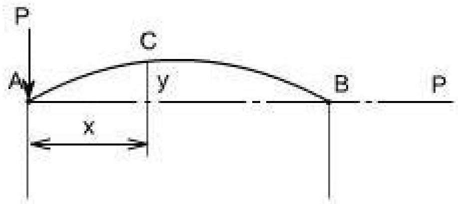
Consider an axially loaded strut, shown below, and is subjected to an axial load 1P' this load 1P' produces a deflection 1y' at a distance 1x' fr om one end.

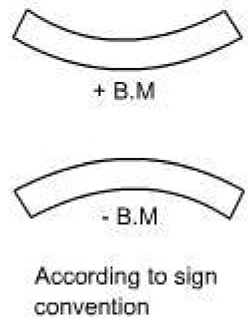
Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.





In this equation 1M' is not a function 1x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Though this equation is in 1y' but we can't say at this stage where the deflection would be maximum or minimum.

So the above differential equation can be arranged in the following form

Let us define a operator

$D = d/dx$

$(D^2 + n^2)\ y = 0$ where $n^2 = P/EI$

This is a second order differential equation which has a solution of the form consisting of com plimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus $y = A \cos(nx) + B \sin(nx)$

Where A and B are some constantss.

Therefore

In order to evaluate the constants A and B let us apply the boundary conditions,

- at x = 0; y = 0
- at x = L ; y = 0

Applying the first boundary condition yields A = 0.

Applying the second boundary con dition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

$$\text{Thus either } B = 0, \text{ or } \sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \text{ or } \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the lea st value of P which will cause the strut to buckle, and it is called the 1 **Euler Crippling Load** 1 P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of irertia

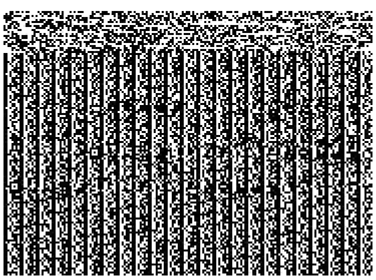
It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than thos e which make sin nL = 0; the strut will remain perfectly straight since

y = B sin nL = 0

For the particular value of



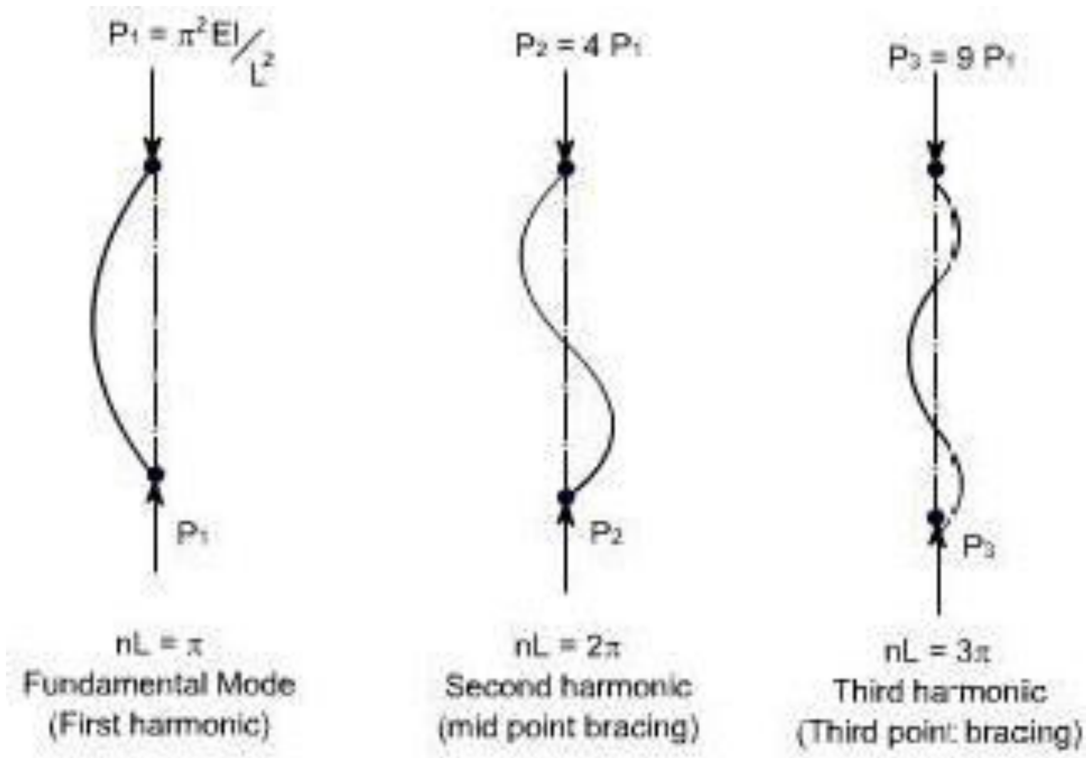
Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 1L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; likewise it will be found that the maximum stress is not proportional to load.

The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi$, 3π, 5π etc are equally valid mathematically and they do, i nfact, produce values of $1P_e$' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

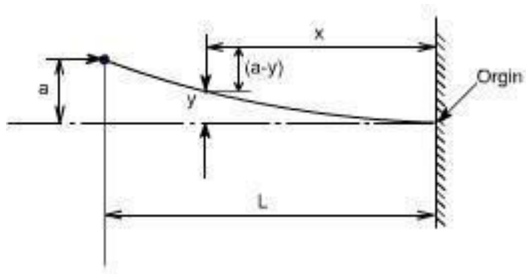
The solution $nL = 2\pi$ produces buc kling in two half 1 waves, 3π in three half-waves etc.



If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundam ental mode and to achieve at least one of the other modes which are theoretically possible. In practical lo ading situations, however, this is rarely acchieved since the high stress associated with the first critic al condition generally ensures immediate collap se.

<u>struts and columns with other en d conditions:</u> Let us consider the struts and columns ha ving different end conditions

Case b: One end fixed and the other free:



writing down the value of bending m oment at the point C

$$B.M|_C = P(a - y)$$

Hence, the differential equation becomes

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

Hence in operator form, the differen tial equation reduces to $(D^2 + n^2) y = n^2 a$

The solution of the above equation would consist of complementary solution and particular s olution, therefore

y_{gen} = A cos(nx) + sin(nx) + P. I

where

P.I = the P.I is a particular value of y which satisfies the differential equation

Hence $y_{P.I}$ = a

Therefore the complete solution becomes

Y = A cos(nx) + B sin(nx) + a

Now imposing the boundary conditions to evaluate the constants A and B

- at x = 0; y = 0

This yields A = -a

- at x = 0; dy/dx =

0 This yields B = 0

Hence

y = −a cos(nx) + a

Futher, at x = L; y = a

Therefore a = - a cos(nx) + a or 0 = cos(nL)

Now the fundamental mode of buck ling in this case would be

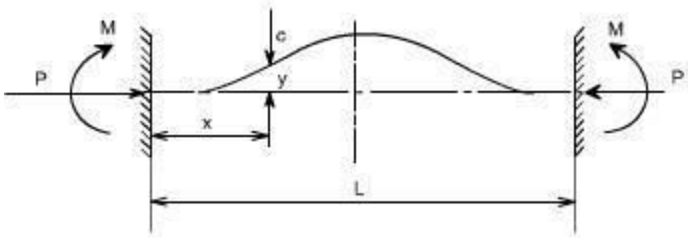
$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}, \text{Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports, since thi s is the property of the support.

Bending Moment at point C = M 1 P.y

$$EI\frac{d^2y}{dx^2} = M - Py$$

or $\frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\ integral}$$

$$y|_{P.I} = \frac{M}{n^2EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\ Cosnx + A\ Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0;y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\ Cos\ nx + \frac{M}{P}$$

$$y = \frac{M}{P}\left(1 - Cos\ nx\right)$$

Futher, it may be noted that at $x = L; y = 0$

Then $0 = \frac{M}{P}\left(1 - Cos\ nL\right)$

Thus, either $\frac{M}{P} = 0$ or $\left(1 - Cos\ nL\right) = 0$

obviously, $\left(1 - Cos\ nL\right) = 0$

$cos\ nL = 1$

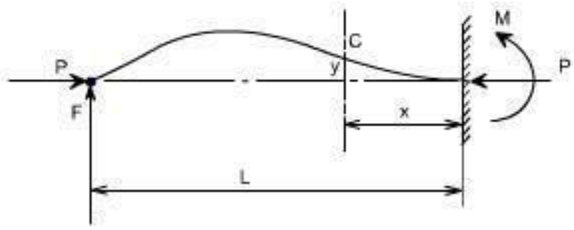
Hence the least solution would be

$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\ L = 2\pi$, Thus, the buckling load or crippling load is

Thus, $$P_e = \frac{4\pi^2.EI}{L^2}$$

Case 4 <u>One end fixed, the other pinned</u>



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pi n. The moment of F about the built in end then balances th e fixing moment.

With the origin at the built in end, th e B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L - x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L - x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L - x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L - x)$$

$$y_{particular} = \frac{F}{n^2EI}(L - x) \text{ or } y = \frac{F}{P}(L - x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L - x)$$

The boundry conditions relevants to the problem are at x=0;y=0

Hence $A = -\frac{FL}{P}$

Also at $x = 0; \frac{dy}{dx} = 0$

Hence $B = \frac{F}{nP}$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L - x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L - x)\right]$$

Also when x = L ; y = 0

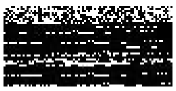
Therefore

nL Cos nL = Sin nL or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.



Where L is the equivalent length of the strut and can be related to the actual length of the str ut depending on the end conditions.

The equivalent length is found to b e the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

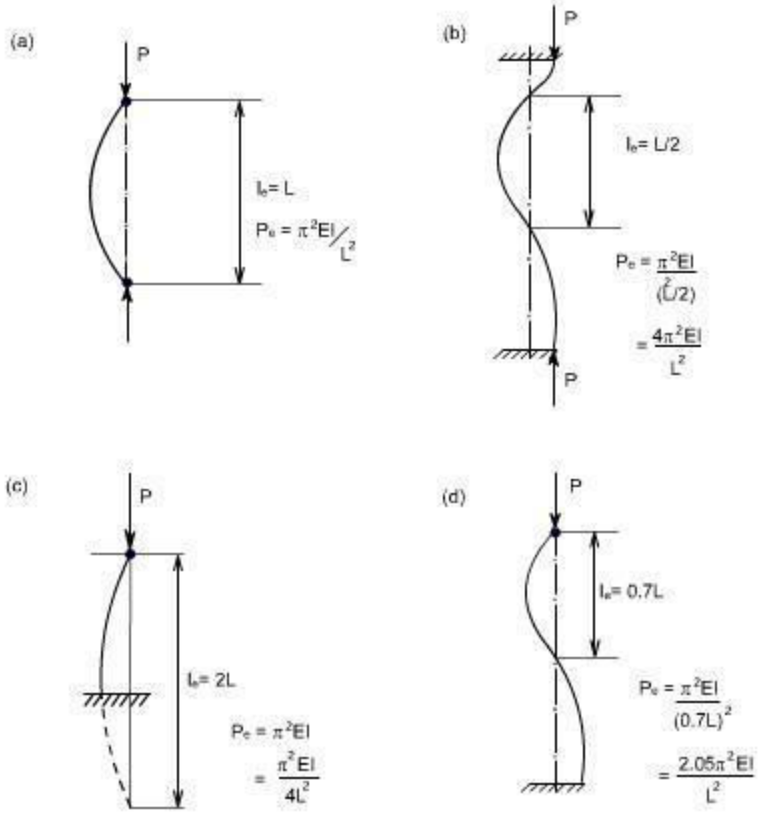
The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we h ave considered so far are:

(a) Both ends pinned (c) One end fixed, other free

(b) Both ends fixed (d) On e end fixed and other pinned



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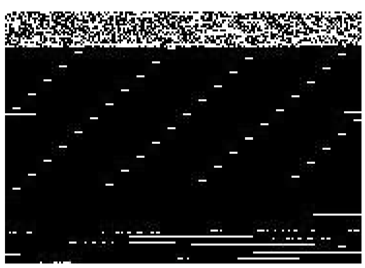
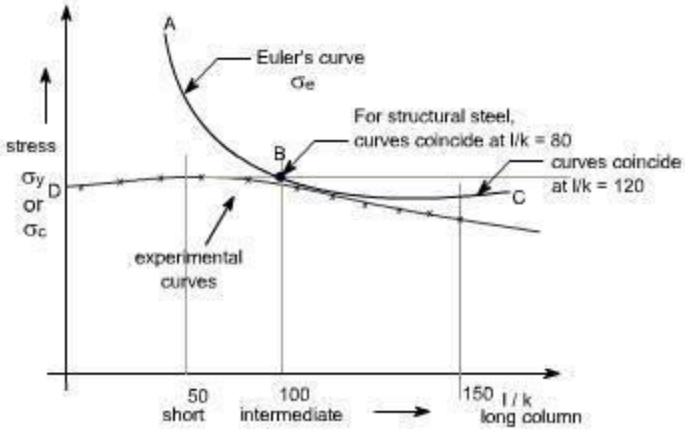


A plot of σ_e versus l / k ratio is shown by the curve ABC.



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[redacted] Where the v alue of index 1n' depends on the material used and the e nd conditions.

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Where P_e = Euler crippling load

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Since the Rankine formulae is a combination of the Euler and crushing load for a strut.

[redacted]

For a very short strut P_e is very large hence 1/ P $_e$would be large so that 1/ P $_e$can be neglected.

Thus P_R = P $_c$, for very large struts, P $_e$ is very small so 1/ P $_e$ would be large and 1/ P $_c$can b e neglected ,hence $P_R = P_e$

The Rankine formulae is therefore valid for extreme values of 1/k.It is also found to be fairly a ccurate for the intermediate values in the range under consideration. Thus rewriting the formula in terms of s tresses, we have

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y}$$

$$\frac{1}{\sigma} = \frac{\sigma_e + \sigma_y}{\sigma_e . \sigma_y}$$

$$\sigma = \frac{\sigma_e . \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{1 + \frac{\sigma_y}{\sigma_e}}$$

For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + a\left(\frac{l}{k}\right)^2}$$

Where  and the value of 1a' is found by conducting experiments on various materi als. Theoretically, but having a value no rmally found by experiment for various materials. This will take into account other types of end conditions.

Therefore 

Typical values of 1a' for use in Ran kine formulae are given below in table.

Material	σ_y or_ c MN/m^2	Value of a	
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note a = 4 x (a for fixed ends)

Since the above values of 1a' are not exactly equal to the theoretical values , the R nkine loads for long struts will not be identical to those estimated by the Euler theory as estimated.

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As we know that the true conditions are never realized , but there are always some imperfections. Let us say that the strut is having s ome initial curvature. i.e., it is not perfectly straight before loading. The situation will influence the stability. Let us analyze this effect.

by a differential calculus

$$R_0 \approx \frac{1}{d^2 y_0 / dx^2} \text{(Approximately)}$$

$$\text{Futher } \frac{E}{R} = \frac{M}{I} \text{ and } \frac{EI}{R} = M$$

$$\text{But for this case } EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = M$$

since strut is having some initial curvature

Now putting

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \text{ and } \frac{1}{R_0} = \frac{d^2 y_0}{dx^2}$$

Where 1 y_0' is the value of deflectio n before the load is applied to the strut when the load is a pplied to the strut the deflection increases to a value 1y'. Hence

$$EI\left[\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right] = M$$

$$EI\frac{d^2y}{dx^2} - EI\frac{d^2y_0}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = M + EI\frac{d^2y_0}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = -Py + EI\frac{d^2y_0}{dx^2}$$

If the pinended strut is under the action of a load P then obviously the BM would be '− py'

Hence

$$EI\frac{d^2y}{dx^2} + Py = EI\frac{d^2y_0}{dx^2}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{d^2y_0}{dx^2}$$

Again letting

$$\frac{P}{EI} = n^2$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2}$$

The initial shape of the strut y_0 may be assumed circular, parabolic or sinusoidal without making much difference to the final results, but the most convenient form is

 where C is some constant or here it is amplitude

Which satisfies the end conditions and corresponds to a maximum deviation 1C'. Any other s hape could be analyzed into a Fourier series of sin e terms. Then

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2} = \frac{d^2}{dx^2}\left[C.\sin\frac{\pi x}{l}\right] = \left(-C.\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right)$$

The computer solution would be therefore be

$$y_{general} = y_{complementry} + y_{PI}$$

$$y = A\cos nx + B\sin nx + \frac{C.\frac{\pi^2}{l^2}}{\left(\frac{\pi^2}{l^2}\right) - n^2}\sin\left(\frac{\pi x}{l}\right)$$

Boundary conditions which are relevant to the problem are

at x = 0 ; y = 0 thus B = 0

Again

when x = l ; y = 0 or x = l / 2 ; dy/dx = 0 the above condition gives B = 0 Therefore the

complete solution would be

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

Again the above solution can be slightly rearranged. since

$$P_e = \frac{\pi^2 EI}{l^2}$$

hence the term $\frac{\frac{\pi^2}{l^2}}{\frac{\pi^2}{l^2} - n^2}$ after multiplying the denominator & numerator by EI is equal to

$$\frac{\frac{\pi^2 EI}{l^2}}{\frac{\pi^2 EI}{l^2} - n^2 EI} = \left[\frac{P_e}{P_e - P}\right]$$

Since $n^2 = \frac{P}{EI}$

where P_e = Euler's load P=applied load

Thus

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

$$y = \left\{\frac{C.P_e}{P_e - P}\right\} \sin\left(\frac{\pi x}{l}\right)$$

The crippling load is again

$$P = P_e = \frac{\pi^2 EI}{l^2}$$

Since the BM for a pin ended strut at any point is given as

M = -Py and

Max BM = P y_{max}

Now in order to define the absolute value in terms of maximum amplitude let us use the sym ol as 1^'.

$$\hat{M} = P.\hat{y}$$

$$= C.\frac{P.P_e}{(P_e - p)}$$

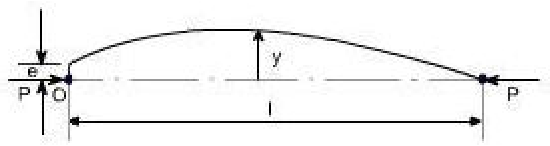
$$\text{Therefore } \hat{M} = \frac{C\ P\ P_e}{[P_e - p]} \text{ since } y_{max^m} = \frac{P_e}{[P_e - p]}$$

$$\sin\frac{\pi x}{l} = 1 \text{ when } \frac{\pi x}{l} = \frac{\pi}{2}$$

$$\text{Hence } \hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$$

Strut with eccentric load

Let 1e' be the eccentricity o f the applied end load, and measuring y from the line of action of the load.



Then 

or $(D^2 + n^2)\ y = 0$ where $n^2 = P / EI$

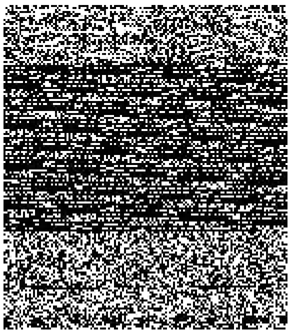
Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions th en we can determine the constants

i.e. at x = 0 ; y = e thus B = e

at x = l / 2 ; dy / dx = 0



Hence the complete solution becomes

$y = A \sin(nx) + B \cos(nx)$

substituting the values of A and B w e get

[redacted]

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = π giving the same crippling load [redacted]. However, due to additional bending moment s et up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x = \frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

$$M_{max^m} = P\ y_{max^m}$$

$$= P e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$$\sigma = \frac{P}{A} + \frac{M}{Z}\ \text{stress due to bending}\ \frac{\sigma}{y} = \frac{M}{I};$$

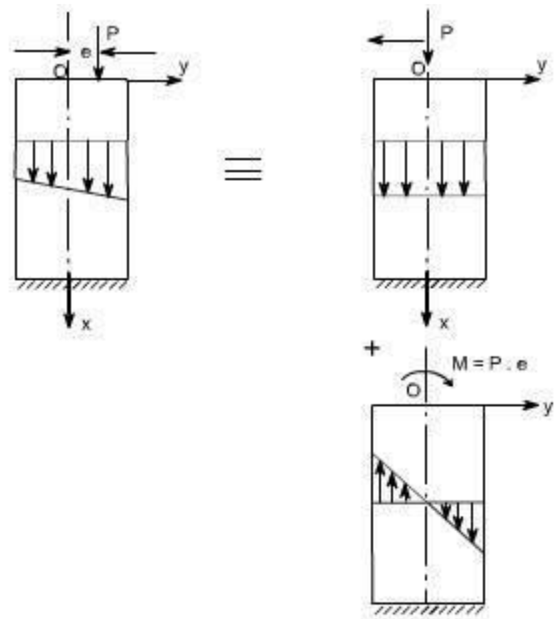
$$M = \sigma\frac{I}{y};\ \sigma_{max} = \frac{M}{Z}\ \text{Wher}\ Z = I/y\ \text{is section modulus}$$

The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be neglible compared with eccentricity 1e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance 1e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compr essive force 1P' and a couple of moment P.e



(vii) The centrally applied load P prod uces a uniform compressive stress over each cross-section as shown by the stress diagram.

(viii) The end moment 1M' produces a linearly varying bending stress as shown in t he figure.

Then by super-impostion, the total compressive stress in any fibre due to combined bending and compression becomes,

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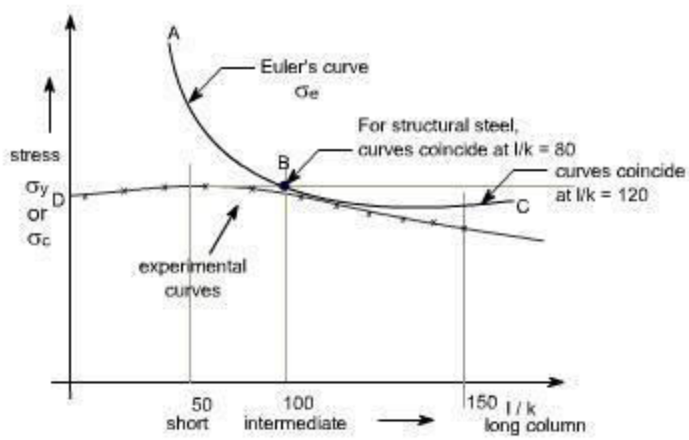
Euler's stress, $\sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$

But, $I = Ak^2$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of s_e versus l / k ratio is shown by the curve ABC.



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$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$ Where the value of index 'n' depends on the material used and the end conditions.

(b) Johnson parabolic formulae : The Johnson parabolic formulae is defined as

$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$ where the value of index 'b' depends on the end conditions.

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$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

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For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

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Where $a = \frac{\sigma_y}{\pi^2 EI}$ and the value of ‘a' is found by conducting experiments on various materials. Theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end conditions.

Therefore $\boxed{\text{Rankine load} = \frac{\sigma_y . A}{1 + a\left(\frac{l}{k}\right)^2}}$

Typical values of ‘a' for use in Rankine formulae are given below in table.

Material	s_y or s_c **MN/m^2**	**Value of a**	
		Pinned ends	**Fixed ends**
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$$\text{Futher } \frac{E}{R} = \frac{M}{I} \text{ and } \frac{EI}{R} = M$$

$$\text{But for this case } EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = M$$

since strut is having some initial curvature

Now putting

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \text{ and } \frac{1}{R_0} = \frac{d^2 y_0}{dx^2}$$

Where ‘ y_0' is the value of deflection before the load is applied to the strut when the load is applied to the strut the deflection increases to a value ‘y'. Hence

$$EI\left[\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right] = M$$

$$EI\frac{d^2y}{dx^2} - EI\frac{d^2y_0}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = M + EI\frac{d^2y_0}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = -Py + EI\frac{d^2y_0}{dx^2}$$

If the pinended strut is under the action of a load P then obviously the BM would be ' – py'

Hence

$$EI\frac{d^2y}{dx^2} + Py = EI\frac{d^2y_0}{dx^2}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{d^2y_0}{dx^2}$$

Again letting

$$\frac{P}{EI} = n^2$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2}$$

The initial shape of the strut y_0 may be assumed circular, parabolic or sinusoidal without making much difference to the final results, but the most convenient form is

$y_0 = C.\sin\frac{\pi x}{l}$ where C is some constant or here it is amplitude

Which satisfies the end conditions and corresponds to a maximum deviation 'C'. Any other shape could be analyzed into a Fourier series of sine terms. Then

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2} = \frac{d^2}{dx^2}\left[C.\sin\frac{\pi x}{l}\right] = \left(-C.\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right)$$

The computer solution would be therefore be

$$y_{general} = y_{complementry} + y_{PI}$$

$$y = A\cos nx + B\sin nx + \frac{C.\frac{\pi^2}{l^2}}{\left(\frac{\pi^2}{l^2}\right) - n^2}\sin\left(\frac{\pi x}{l}\right)$$

Boundary conditions which are relevant to the problem are

at x = 0 ; y = 0 thus B = 0

Again

when x = l ; y = 0 or x = l / 2 ; dy/dx = 0

the above condition gives B = 0

Therefore the complete solution would be

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

Again the above solution can be slightly rearranged.since

$$P_e = \frac{\pi^2 EI}{l^2}$$

hence the term $\frac{\frac{\pi^2}{l^2}}{\frac{\pi^2}{l^2} - n^2}$ after multiplying the denominator & numerator by EI is equal to

$$\frac{\frac{\pi^2 EI}{l^2}}{\frac{\pi^2 EI}{l^2} - n^2 EI} = \left[\frac{P_e}{P_e - P}\right]$$

Since $n^2 = \frac{P}{EI}$

where P_e = Euler's load P=applied load

Thus

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

$$y = \left\{\frac{C.P_e}{P_e - P}\right\} \sin\left(\frac{\pi x}{l}\right)$$

The crippling load is again

$$P = P_e = \frac{\pi^2 EI}{l^2}$$

Since the BM for a pin ended strut at any point is given

as M = -Py and

Max BM = P y_{max}

Now in order to define the absolute value in terms of maximum amplitude let us use the symbol as '^'.

$$\hat{M} = P.\hat{y}$$
$$= C.\frac{P.P_e}{(P_e - p)}$$

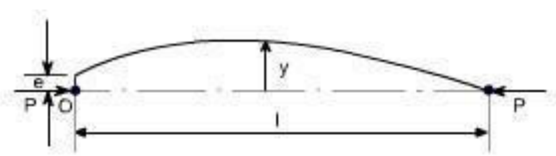
Therefore $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$ since $y_{max^m} = \frac{P_e}{[P_e - p]}$

$\sin\frac{\pi x}{l} = 1$ when $\frac{\pi x}{l} = \frac{\pi}{2}$

Hence $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$

Strut with eccentric load

Let 'e' be the eccentricity of the applied end load, and measuring y from the line of action of the load.



Then $E\,I\,\frac{d^2 y}{dx^2} = -\,P\,y$

or $(D^2 + n^2)\,y = 0$ where $n^2 = P / EI$

Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions then we can determine the constants i.e.

at $x = 0$; $y = e$ thus $B = e$

at $x = l / 2$; $dy / dx = 0$

Therefore

$$A \cos \frac{nl}{2} - B \sin \frac{nl}{2} = C$$

$$A \cos \frac{nl}{2} = B \sin \frac{nl}{2}$$

$$A = B \tan \frac{nl}{2}$$

$$A = e \tan \frac{nl}{2}$$

Hence the complete solution becomes

$y = A \sin(nx) + B \cos(nx)$

substituting the values of A and B we get

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = p giving the same crippling load $P_e = \frac{\pi^2 EI}{l^2}$. However, due to additional bending moment set up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x=\frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

$$M_{max^m} = P\ y_{max^m}$$

$$= P\,e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$$\sigma = \frac{P}{A} + \frac{M}{Z} \text{ stress due to bending } \frac{\sigma}{y} = \frac{M}{I};$$

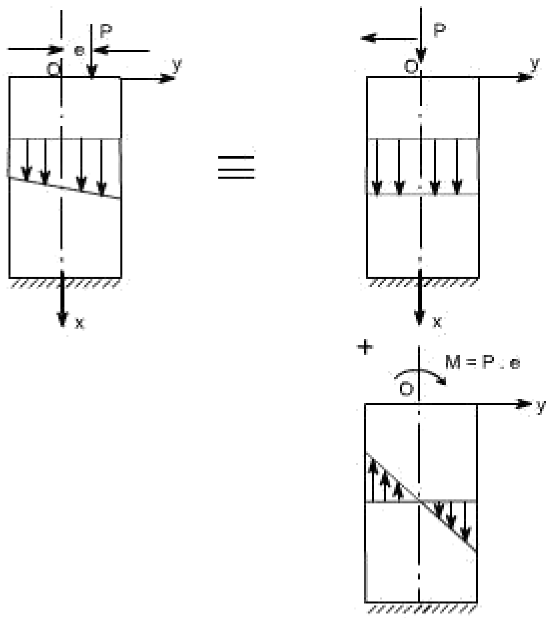
$$M = \sigma\frac{I}{y};\ \sigma_{max} = \frac{M}{Z} \text{ Wher } Z = I/y \text{ is section modulus}$$

The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be neglible compared with eccentricity ‘e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance ‘e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compressive force ‘P' and a couple of moment P.e



The centrally applied load P produces a uniform compressive $\sigma_1 = \frac{P}{A}$ stress over each cross-section as shown by the stress diagram.

(vi) The end moment 'M' produces a linearly varying bending $\sigma_2 = \frac{My}{I}$ as shown in the figure.

Then by super-impostion, the total compressive stress in any fibre due to combined bending and compression becomes,

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

$$\sigma = \frac{P}{A} + \frac{M}{I/y}$$

$$\boxed{\sigma = \frac{P}{A} + \frac{M}{Z}}$$

Chapter 3

BEAM COLUMNS & DIRECT & BENDING STRESSES

Preamble

Engineering science is usually subdivided into number of topics such as

(vii) Solid Mechanics

(viii) Fluid Mechanics

(ix)Heat Transfer

(x) Properties of materials and soon Although there are close links between them in terms of the physical principles involved and methods of analysis employed.

The solid mechanics as a subject may be defined as a branch of applied mechanics that deals with behaviours of solid bodies subjected to various types of loadings. This is usually subdivided into further two streams i.e Mechanics of rigid bodies or simply Mechanics and Mechanics of deformable solids.

The mechanics of deformable solids which is branch of applied mechanics is known by several names i.e. strength of materials, mechanics of materials etc.

Mechanics of rigid bodies:

The mechanics of rigid bodies is primarily concerned with the static and dynamic behaviour under external forces of engineering components and systems which are treated as infinitely strong and undeformable Primarily we deal here with the forces and motions associated with particles and rigid bodies.

Mechanics of deformable solids :

Mechanics of solids:

The mechanics of deformable solids is more concerned with the internal forces and associated changes in the geometry of the components involved. Of particular importance are the properties of the materials used, the strength of which will determine whether the components fail by breaking in service, and the stiffness of which will determine whether the amount of deformation they suffer is acceptable. Therefore, the subject of mechanics of materials or strength of materials is central to the whole activity of engineering design. Usually the objectives in analysis here will be the determination of the stresses, strains, and deflections produced by loads. Theoretical analyses and experimental results have an equal roles in this field.

Analysis of stress and strain :

Concept of stress : Let us introduce the concept of stress as we know that the main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

The externally applied forces are termed as loads. These externally applied forces may be due to any one of the reason.

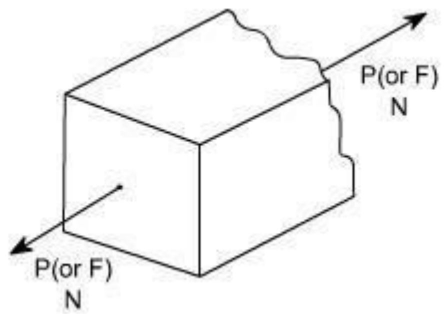
- due to service conditions
- due to environment in which the component works

- through contact with other mem bers
- due to fluid pressures
- due to gravity or inertia forces.

As we know that in mechanics of d eformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or react d by internal forces which are set up within the particles of material due to cohesion.

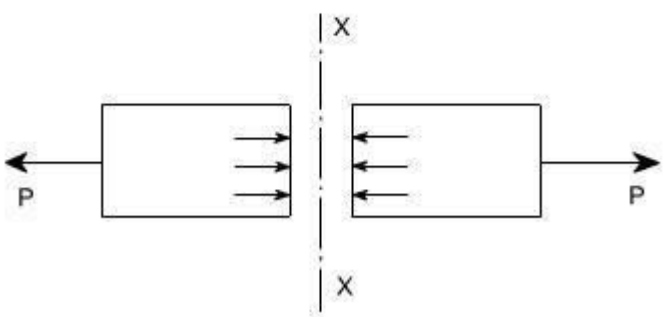
These internal forces give rise to a concept of stress. Therefore, let us define a stress Therefore, let us define a term stress

Stress:



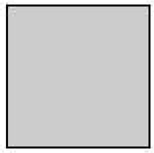
Let us consider a rectangular bar of some cross 1 sectional area and subjected to some load or force (in Newtons)

Let us imagine that the same recta ngular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown



Now stress is defined as the force intensity or force per unit area. Here we use a symbol σ to represent the stress.

Where A is the area of the X 1 section



Here we are using an assumption that the total force or total load carried by the rectangular b ar is uniformly distributed over its cross 1 section.

But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations.

If the force carried by a component is not uniformly distributed over its cross 1 sectional area, A, we must consider a small area, 1δA' which c arries a small load δP, of the total force 1P', Then definition of stress is

As a particular stress generally hol ds true only at a point, therefore it is defined mathematically as

Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa)

MPa = 10^6 Pa

GPa = 10^9 Pa

KPa = 10^3 Pa

Some times N / mm^2 units are also used, because this is an equivalent to MPa. While US cu stomary unit is pound per square inch psi.

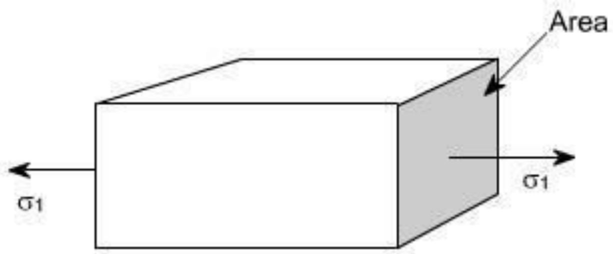
TYPES OF STRESSES :

only two basic stresses exists : (1) normal stress and (2) shear shear stress. Other stresses either are similar to these basic stresses or ar e a combination of these e.g. bending stress is a combina tion tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress.

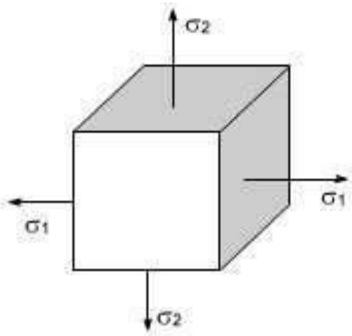
Let us define the normal stresses and shear stresses in the following sections.

Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally de noted by a

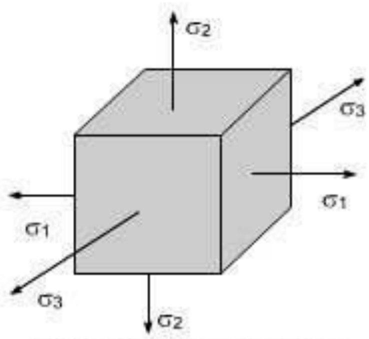
Greek letter (σ)



This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :



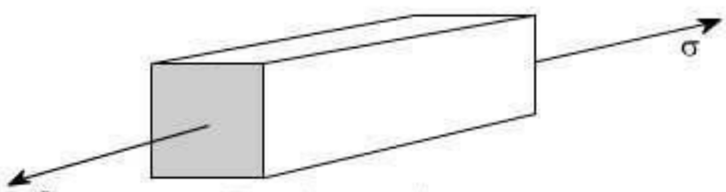
(Biaxial state of stress)



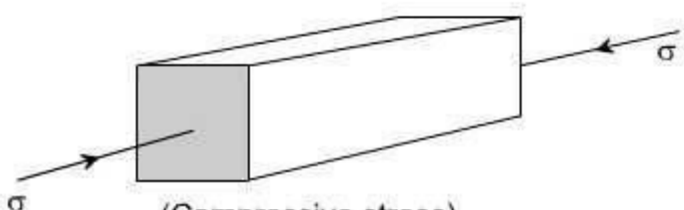
(Triaxial state of stress)

Tensile or compressive stresses :

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

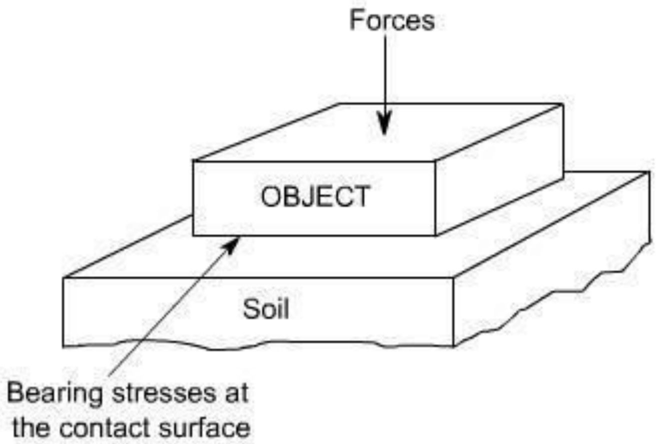


(Tensile stress)



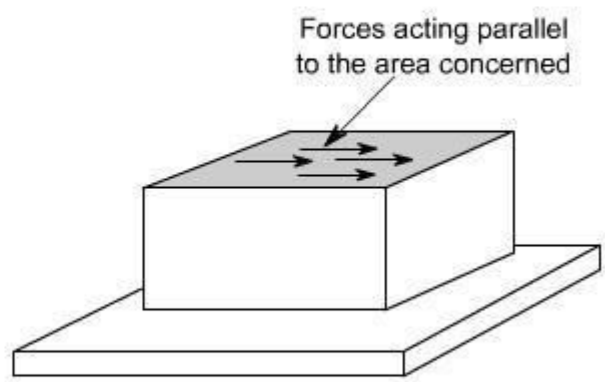
(Compressive stress)

Bearing Stress : When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



Shear stresses :

Let us consider now the situation, w here the cross 1 sectional area of a block of material is subject to a distribution of forces which are para llel, rather than normal, to the area concerned. Such forc es are associated with a shearing of the m aterial, and are referred to as shear forces. The resulting force interistes are known as shear stresses.



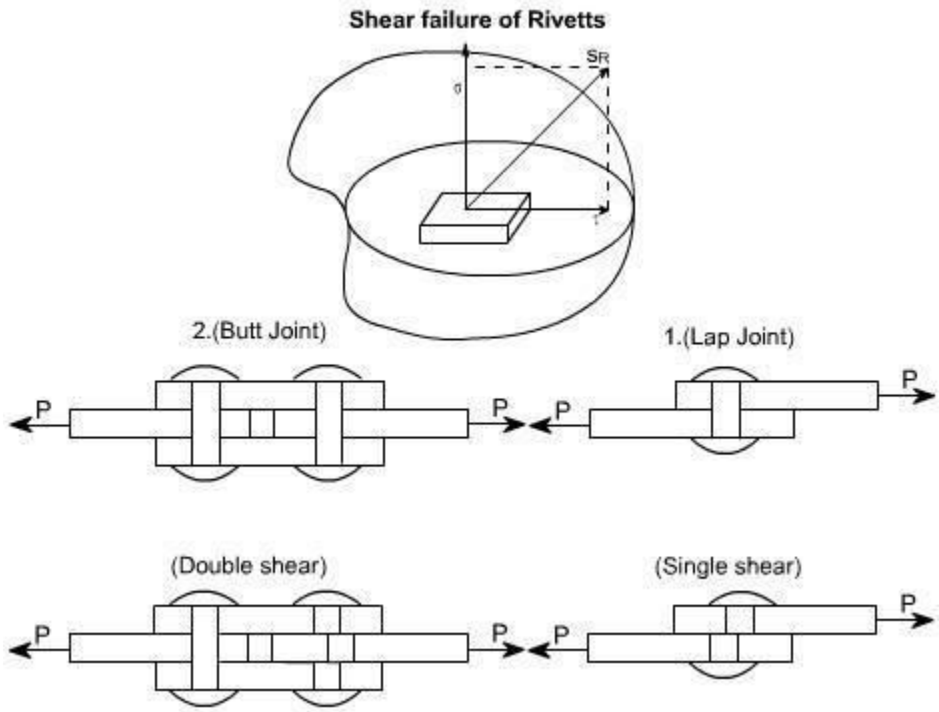
The resulting force intensities are known as shear stresses, the mean shear stress being equ al to

Where P is the total force and A the area over which it acts.

As we know that the particular stresss generally holds good only at a point therefore we can define shear stress at a point as

The greek symbol τ (tau) (sugges ting tangential) is used to denote shear stress.

However, it must be borne in mind that the stress (resultant stress) at any point in a body is basically resolved into two components σ and τone acts perpendicular and other parallel to the area concerned, as it is clearly defined in the following figure.

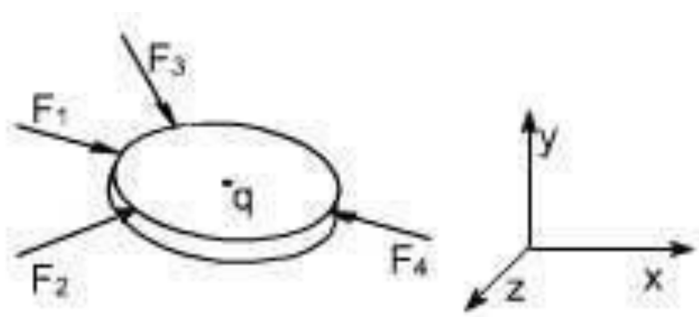


The single shear takes place on the single plane and the shear area is the cross - sectional of the rivett, whereas the double shear takes place in the case of Butt joints of rivetts and the shear area is the twice of the X - sectional area of the rivett.

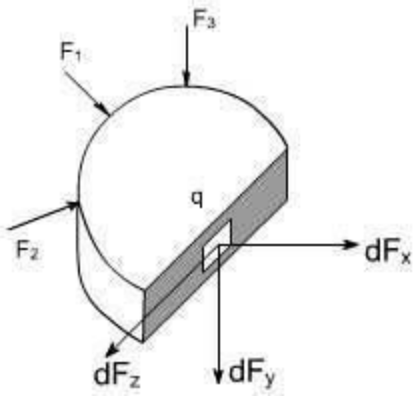
ANALYSIS OF STERSSES

General State of stress at a point :

Stress at a point in a material body has been defined as a force per unit area. But this definition is some what ambiguous since it depends upon what area we consider at that point. Let us, consider a point 1q' in the interior of the body



Let us pass a cutting plane through a pont 'q' perpendicular to the x - axis as shown below



The corresponding force components can be shown like this

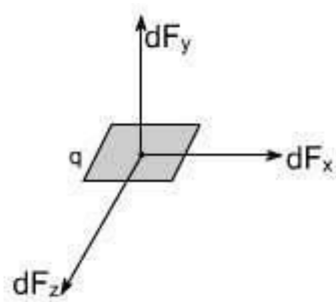
$dF_x = \sigma_{xx}. da_x$

$dF_y = \tau_{xy}. da_x$

$dF_z = \tau_{xz}. da_x$

where da_x is the area surrounding the point 'q' when the cutting plane ⊥ r is to x - axis.

In a similar way it can be assummed that the cutting plane is passed through the point 'q' perpendicular to the y - axis. The corresponding force components are shown below



The corresponding force components may be written

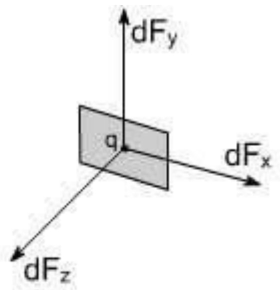
as $dF_x = \tau_{yx}. da_y$

$dF_y = \sigma_{yy}. da_y$

$dF_z = \tau_{yz}. da_y$

where da_y is the area surrounding the point 'q' when the cutting plane ⊥ r is to y - axis.

In the last it can be considered that the cutting plane is passed through the point 'q' perpendicular to the z - axis.



The corresponding force components may be written as

$dF_x = \tau_{zx}.\ da_z$

$dF_y = \tau_{zy}.\ da_z$

$dF_z = \sigma_{zz}.\ da_z$

where da_z is the area surrounding the point 'q' when the cutting plane $\perp^r$ is to z - axis.

Thus, from the foregoing discussion it is amply clear that there is nothing like stress at a point 'q' rather we have a situation where it is a combination of state of stress at a point q. Thus, it becomes imperative to understand the term state of stress at a point 'q'. Therefore, it becomes easy to express astate of stress by the scheme as discussed earlier, where the stresses on the three mutually perpendiclar planes are labelled in the manner as shown earlier. the state of stress as depicted earlier is called the general or a triaxial state of stress that can exist at any interior point of a loaded body.

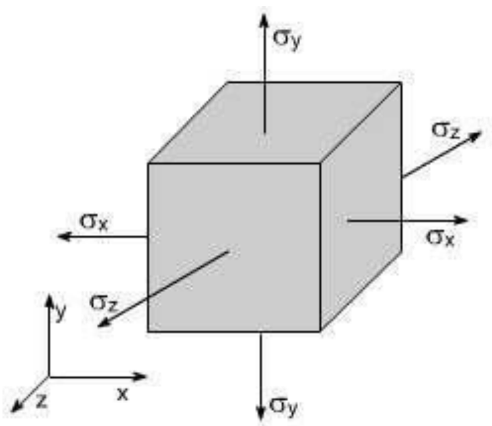
Before defining the general state of stress at a point. Let us make overselves conversant with the notations for the stresses.

We have already chosen to distinguish between normal and shear stress with the help of symbols σ and τ .

Cartesian - co-ordinate system

In the Cartesian co-ordinates system, we make use of the axes, X, Y and Z

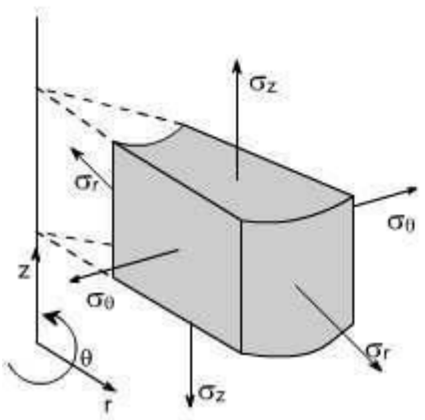
Let us consider the small element of the material and show the various normal stresses acting the faces



Thus, in the Cartesian co-ordinates system the normal stresses have been represented by σ_x, σ_y and σ_z.

Cylindrical - co-ordinate system

In the Cylindrical - co-ordinate system we make use of co-ordinates r, θ and Z.



Thus, in the Cylindrical co-ordinates system, the normal stresses i.e components acting over a element is being denoted by σ_r, σ_θ and σ_z.

Sign convention : The tensile forces are termed as (+ve) while the compressive forces are termed as negative (-ve).

First sub script : it indicates the direction of the normal to the surface.

Second subscript : it indicates the direction of the stress.

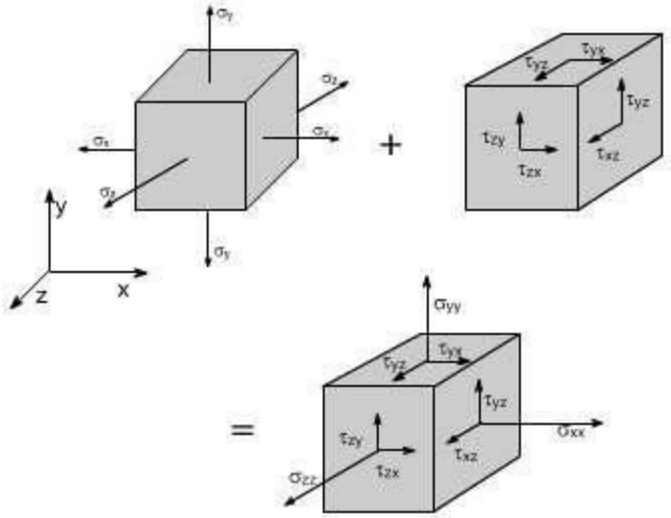
It may be noted that in the case of normal stresses the double script notation may be dispensed with as the direction of the normal stress and the direction of normal to the surface of the element on which it acts is the same. Therefore, a single subscript notation as used is sufficient to define the normal stresses.

Shear Stresses : With shear stress components, the single subscript notation is not practical, because such stresses are in direction parallel to the surfaces on which they act. We therefore have two directions to specify, that of normal to the surface and the stress itself. To do this, we stress itself. To do this, we attach two subscripts to the symbol ' τ' , for shear stresses.

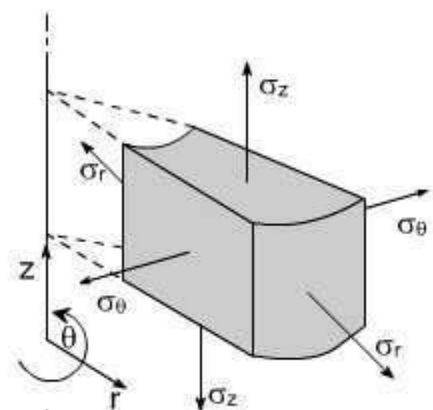
In cartesian and polar co-ordinates, we have the stress components as shown in the figures.

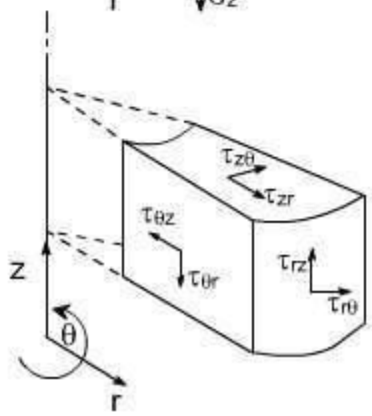
τ_{xy} , τ_{yx} , τ_{yz} , τ_{zy} , τ_{zx} , τ_{xz}

$\tau_{r\theta}$, $\tau_{\theta r}$, $\tau_{\theta z}$, $\tau_{z\theta}$, τ_{zr} , τ_{rz}

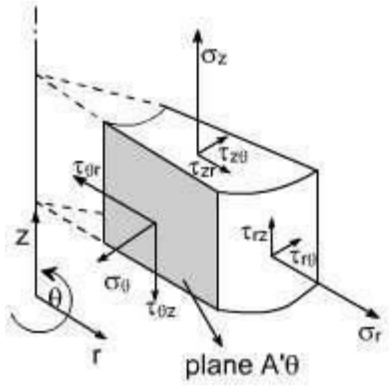


So as shown above, the normal stresses and shear stress components indicated on a small element of material seperately has been combined and depicted on a single element. Similarly for a cylindrical co-ordinate system let us shown the normal and shear stresses components separately.





Now let us combine the normal and shear stress components as shown below :



Now let us define the state of stress at a point formally.

State of stress at a point :

By state of stress at a point, we mean an information which is required at that point such that it remains under equilibrium. or simply a general state of stress at a point involves all the normal stress components, together with all the shear stress components as shown in earlier figures.

Therefore, we need nine components, to define the state of stress at a point

$\sigma_x \; \tau_{xy} \; \tau_{xz}$

$\sigma_y \; \tau_{yx} \; \tau_{yz}$

$\sigma_z \tau_{zx} \tau_{zy}$

If we apply the conditions of equilibrium which are as follows:

$\Sigma F_x = 0 \; ; \; \Sigma M_x = 0$

$\Sigma F_y = 0 \; ; \; \Sigma M_y = 0$

$\Sigma F_z = 0 \; ; \; \Sigma M_z = 0$

Then we get

$\tau_{xy} = \tau_{yx}$

$\tau_{yz} = \tau_{zy}$

$\tau_{zx} = \tau_{xy}$

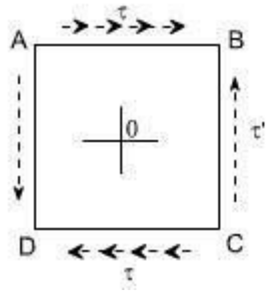
Then we will need only six components to specify the state of stress at a point i.e

$\sigma_x , \sigma_y , \sigma_z , \tau_{xy} , \tau_{yz} , \tau_{zx}$

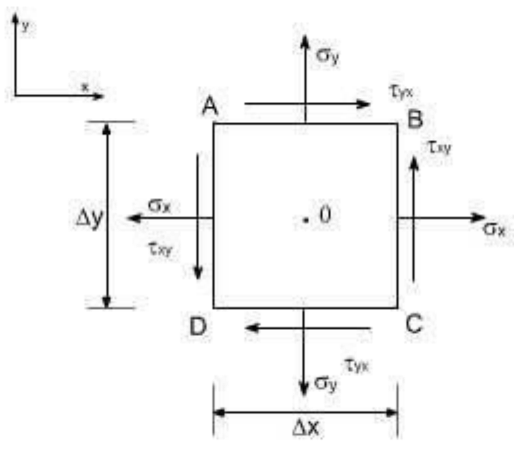
Now let us define the concept of complementary shear stresses.

Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium.



on planes AB and CD, the shear stress τ acts. To maintain the static equilibrium of this element, on planes AD and BC, τ' should act, we shall see that τ' which is known as the complementary shear stress would come out to equal and opposite to the_ _. Let us prove this thing for a general case as discussed below:



The figure shows a small rectangular element with sides of length x, y parallel to x and y directions. Its thickness normal to the plane of paper is z in z 1 direction. All nine normal and shear stress components may act on the element, only those in x and y directions are shown.

Sign convections for shear stresses:

Direct stresses or normal stresses

=tensile +ve

=compressive 1ve

Shear stresses:

=tending to turn the element C.W +ve.

=tending to turn the element C.C.W 1 ve.

The resulting forces applied to the element are in equilibrium in x and y direction. (Although other normal and shear stress components are n ot shown, their presence does not affect the final conclusion).

Assumption : The weight of the el ement is neglected.

Since the element is a static piece of solid body, the moments applied to it must also be in eq uilibrium. Let 1O' be the centre of the element. Let us consider the axis through the point 1O'. the resultant force associated with normal stresses σ_x and σ_y acting on the sides of the element each pass throu gh this axis, and therefore, have no moment.

Now forces on top and bottom surf aces produce a couple which must be balanced by the for ces on left and right hand faces

Thus,

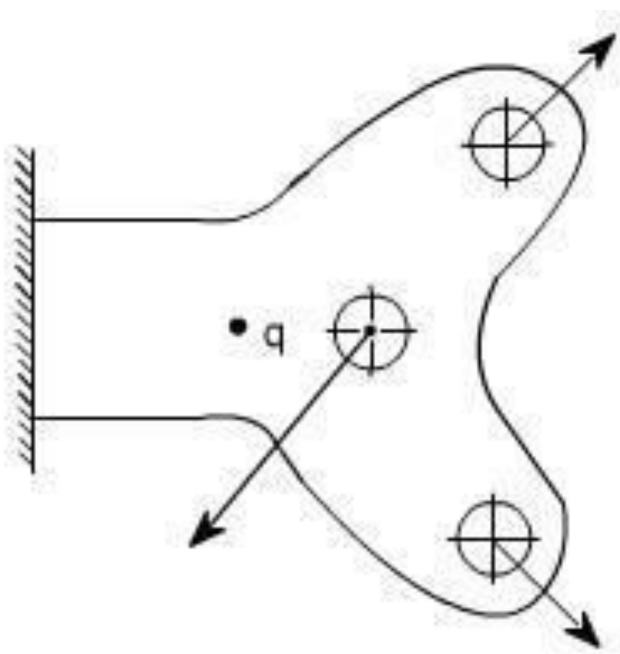
$\tau_{yx} . x . z . y = \tau_{xy} . x . z . y$

[illegible]

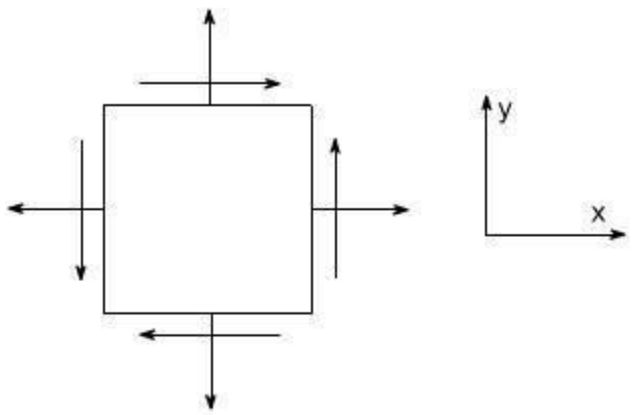
In other word, the complementary s hear stresses are equal in magnitude. The same form of r elationship can be obtained for the other two pair of shear stress components to arrive at the relations

[illegible]

Analysis of Stresses:



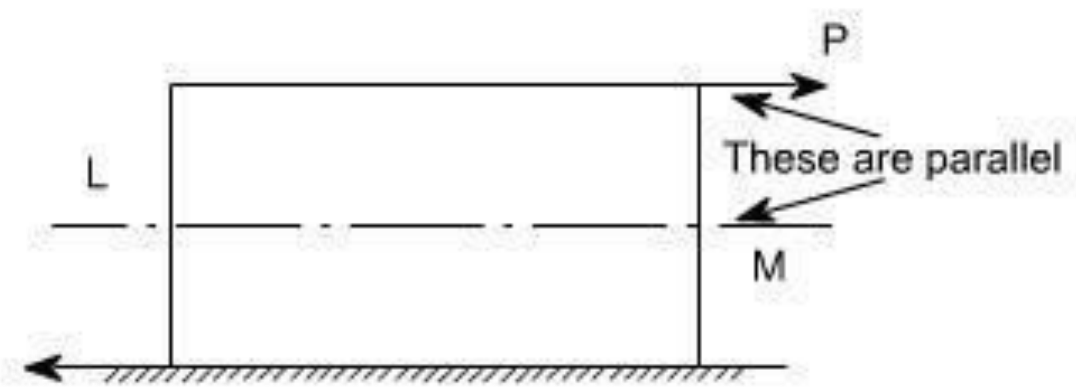
Consider a point 1q' in some sort of structural member like as shown in figure below. Assuming that at point exist. 1q' a plane state of stress exist. i.e. the state of state stress is to describe by a parameters σ_x, σ_y and τ_{xy} These stresses could be indicate a on the two dimensional diagram as shown below:



This is a commen way of representing the stresses. It must be realize a that the material is unaware of what we have called the x and y axes. i.e. the material has to resist the loads irrespective less of how we wish to name them or whether they are horizontal, vertical or otherwise further more, the material will fail when the stresses exceed beyond a permissible value. Thus, a fundamental problem in engineering design is to determine the maximum normal stress or maximum shear stress at any particular point in a body.

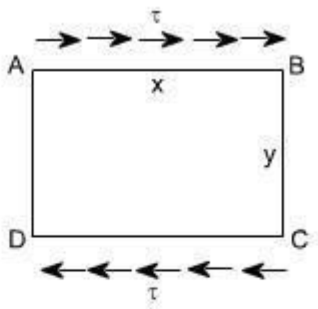
There is no reason to believe apriori that σ_x, σ_y and τ_{xy} are the maximum value. Rather the maximum stresses may associates themselves with some other planes located at 1θ'. Thus, it becomes imperative to determine the values of σ_θ and_ $_\theta$. In order tto achieve this let us consider the following.

Shear stress:



If the applied load P consists of two equal and opposite parallel forces not in the same line, than there is a tendency for one part of the body to slide over or shear from the other part across any section LM. If the cross section at LM measured parallel to the load is A, then the average value of shear stress $\tau = P/A$. The shear stress is tangential to the area over which it acts.

If the shear stress varies then at a point then τ may be defined as 



Complementary shear stress:

Let ABCD be a small rectangular element of sides x, y and z perpendicular to the plane of paper let there be shear stress acting on planes AB and CD

It is obvious that these stresses will from a couple (τ . xz)y which can only be balanced by t ngential forces on planes AD and BC. These are known as complementary shear stresses. i.e. the existence of shear stresses on sides AB and CD of the element implies that there must also be complementary shear stresses on to maintain equilibrium.

Let τ' be the complementary shear stress induced on planes

AD and BC. Then for the equilibriu m (_ . xz)y = τ' (yz)x

$$\tau = \tau'$$

Thus, every shear stress is accomp anied by an equal complementary shear stress.

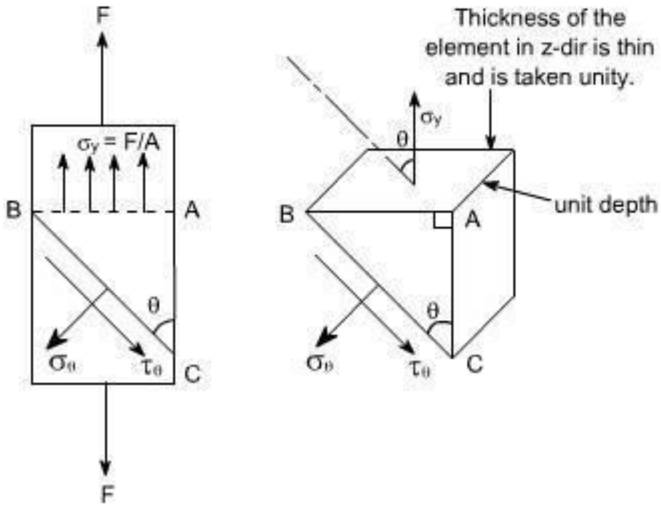
Stresses on oblique plane: Till no w we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor ta ngential to the plane.

A plane stse of stress is a 2 dimens ional stae of stress in a sense that the stress components in one direction are all zero i.e

$\sigma_z = \tau_{yz} = \tau_{zx} = 0$

examples of plane state of stress in cludes plates and shells.

Consider the general case of a bar under direct load F giving rise to a stress σ_y vertically



The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point.

The stresses change with the inclination of the planes passing through that point i.e. the stre ss on the faces of the element vary as the angular position of the element changes.

Let the block be of unit depth now c onsidering the equilibrium of forces on the triangle portion ABC

Resolving forces perpendicular to B C, gives

$\sigma_\theta . BC . 1 = \sigma_y \sin\theta \; . \; AB \; . \; 1$

but AB/BC = sinθ or AB = BCsinθ

Substituting this value in the above equation, we get

$\sigma_\theta . BC . 1 = \sigma_y \sin\theta \; . \; BC \sin\theta \; . \; 1$ or [redacted] (1)

Now resolving the forces parallell to BC

$\tau_\theta . BC . 1 = \sigma_y \cos\theta \; . \; AB \sin\theta \; . \; 1$

again AB = BCcosθ

$\tau_\theta . BC . 1 = \sigma_y \cos\theta \; . \; BC \sin\theta \; . \; 1$ or_ $_\theta = \sigma_y \sin\theta \cos\theta$

[redacted] (2)

If $\theta = 90^0$ the BC will be parallel to AB and $\tau_\theta = 0$, i.e. there will be only direct stress or normal stress.

By examining the equations (1) and (2), the following conclusions may be drawn

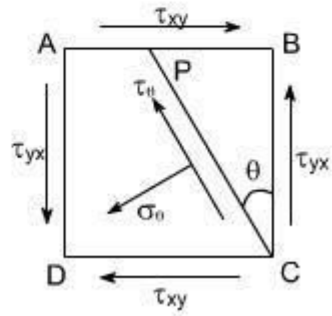
(iii) The value of direct stress σ_θ is maximum and is equal to σ_y when $\theta = 90^0$.

(iv) The shear stress τ_θ has a maximum value of 0.5 σ_y when $\theta = 45^0$

(v) The stresses σ_θ and σ_θ are not simply the resolution of σ_y

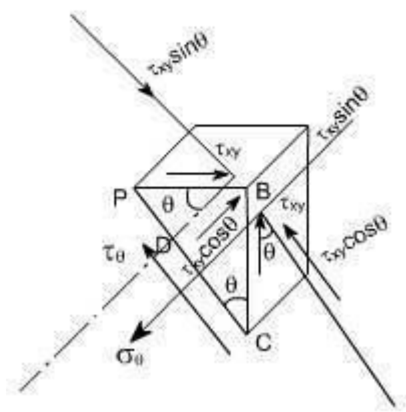
Material subjected to pure shear:

Consider the element shown to which shear stresses have been applied to the sides AB and DC



Complementary shear stresses of equal value but of opposite effect are then set up on the sides AD and BC in order to prevent the rotation of the element. Since the applied and complementary shear stresses are of equal value on the x and y planes. Therefore, they are both represented by the symbol τ_{xy}.

Now consider the equilibrium of portion of PBC



Assuming unit depth and resolving normal to PC or in the direction of σ_θ

$$\sigma_\theta.PC.1 = \_\,_{xy}.PB.\cos\theta.1 + \_\,_{xy}.BC.\sin\theta.1$$

$$= \tau_{xy}.PB.\cos\theta + \tau_{xy}.BC.\sin\theta$$

Now writing PB and BC in terms of PC so that it cancels out from the two sides

$$PB/PC = \sin\theta \quad BC/PC = \cos\theta$$

$\sigma_\theta.PC.1 = \tau_{xy}.\cos\theta\sin\theta PC + \tau_{xy}.\cos\theta.\sin\theta PC$

$$\sigma_\theta = 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_\theta = \tau_{xy}.2.\sin\theta\cos\theta$$

[illegible] (1)

Now resolving forces parallel to PC or in the direction τ_θ.then $\tau_{xy}PC\ .\ 1 =\_\ _{xy}\ .\ PB\sin\theta - \tau_{xy}\ .\ B\ C\cos\theta$

$-$ve sign has been put because this component is in the same direction as that of τ_θ.

again converting the various quantities in terms of PC we have

$\tau_{xy}PC\ .\ 1 =\_\ _{xy}\ .\ PB.\sin^2\theta - \tau_{xy}\ .\ PC\cos^2\theta$

$= -[\_\ _{xy}(\cos^2\theta\ \_\sin^2\theta)\ ]$

$= -\tau_{xy}\cos 2\ _$or$_$ [illegible] (2)

the negative sign means that the se nse of τ_θ is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively

From equation (1) i.e,

$\sigma_\theta = \tau_{xy}\sin 2\theta$

The equation (1) represents that th e maximum value of σ_θ is$_\ _{xy}$ when $\theta = 45^0$.

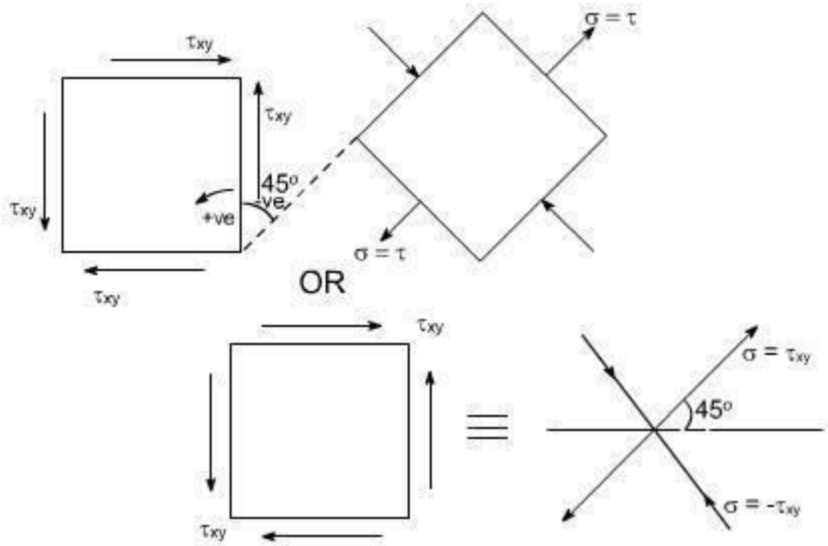
Let us take into consideration the equation (2) which states that

$\tau_\theta =\_\ \tau_{xy}\cos 2\theta$

It indicates that the maximum value of τ_θ is$_\ _{xy}$ when $\theta = 0^0$ or 90^0. it has a value zero when $\theta = 45^0$.

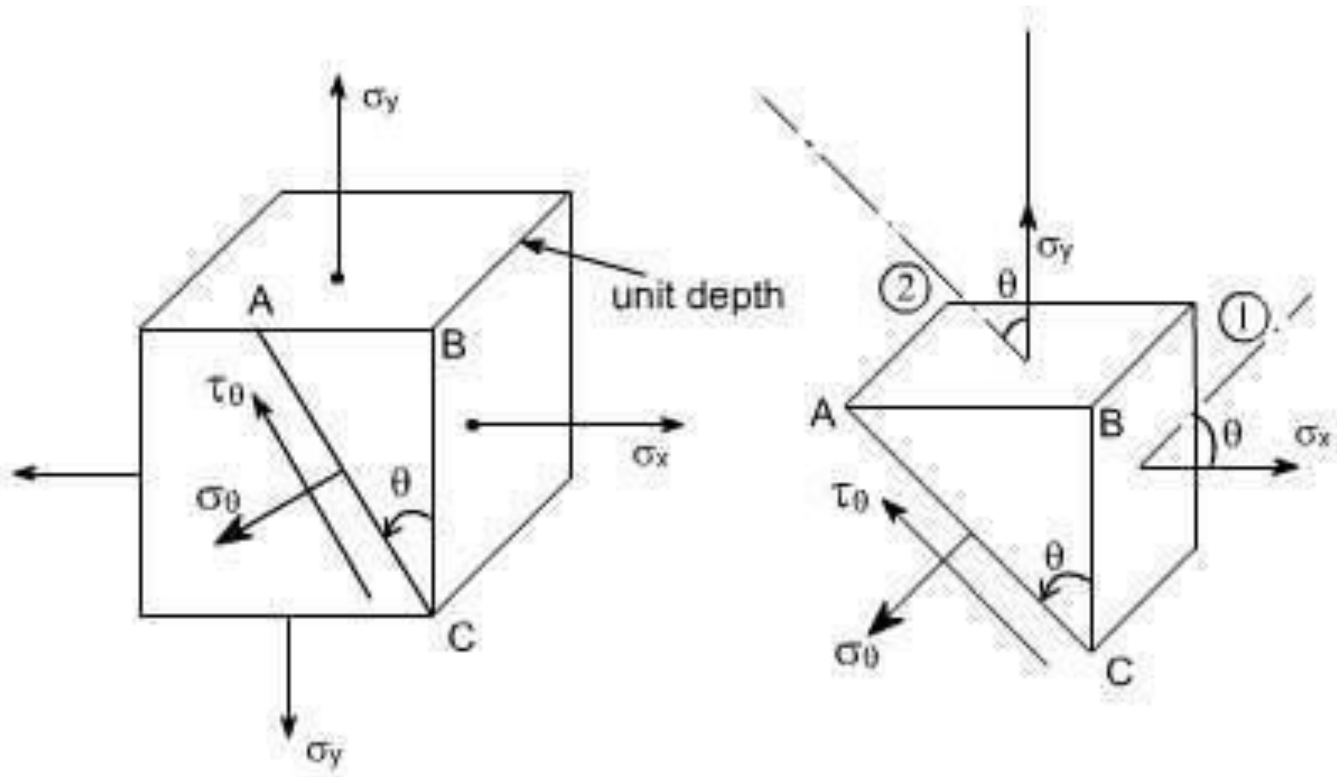
From equation (1) it may be noticed that the normal component σ_θ has maximum and minimu m values of $+\tau_{xy}$ (tension) and $-\tau_{xy}$(compression) on plane at $\pm 45^0$ to the applied shear and on these plan es the tangential component τ_θ is zero.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile each located at 45^0 to the original shear directions as depicted in the figure below:



Material subjected to two mutually perpendicular direct stresses:

Now consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, σ_x and σ_y acting right angles to each other.



for equilibrium of the portion ABC, resolving perpendicular to AC

σ_θ . AC.1 = $\sigma_y \sin\theta$. AB.1 + $\sigma_x \cos\theta$. BC.1

converting AB and BC in terms of A C so that AC cancels out from the sides

$\sigma_\theta = \sigma_y \sin^2\theta + \sigma_x \cos^2\theta$

Futher, recalling that $\cos^2\theta$ _$\sin^2\theta = \cos 2\theta$ or $(1 - \cos 2\theta)/2 = \sin^2\theta$

Similarly $(1 + \cos 2\theta)/2 = \cos^2 q$

Hence by these transformations the expression for σ_θ reduces to

$= 1/2\sigma_y(1 - \cos 2\theta) + 1/2\sigma_x(1 + \cos 2\theta)$

On rearranging the various terms w e get

(3)

Now resolving parallal to AC

s_q.AC.1= $-\tau_{xy}$..cosθ.AB.1+_ $_{xy}$.BC.s inθ.1

The 1 ve sign appears because this component is in the same direction as that of AC.

Again converting the various quantities in terms of AC so that the AC cancels out from the two sides.

(4)

Conclusions :

The following conclusions may be d rawn from equation (3) and (4)

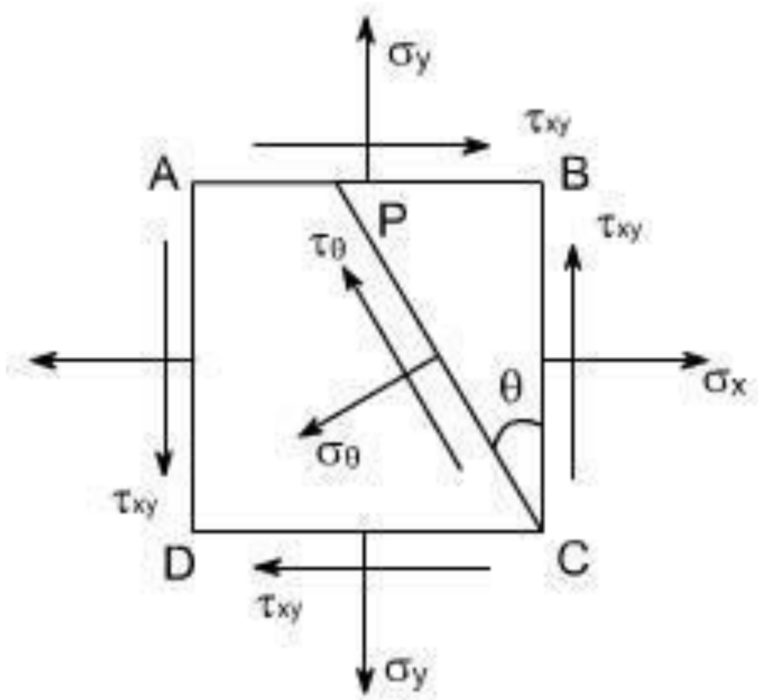
(f) The maximum direct stress wo uld be equal to σ_x or σ_y which ever is the greater, when $\theta = 0^0$ or 90^0

(g) The maximum shear stress in t he plane of the applied stresses occurs when _= 45^0

Material subjected to combined direct and shear stresses:

Now consider a complex stress system shown below, acting on an element of material.

The stresses σ_x and σ_y may be com pressive or tensile and may be the result of direct forces or as a result of bending.The shear stresses may b e as shown or completely reversed and occur as a result o f either shear force or torsion as shown in the figu re below:



As per the double subscript notatio n the shear stress on the face BC should be notified as τ_{yx} , however, we have already seen that for a pair of shear stresses there is a set of complementary shear stre sses generated such that $\tau_{yx} = \tau_{xy}$

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure stae o f stress shear. In this case the various formulas deserved are as follows

$\sigma_\theta = \tau_{yx}$ sin2_

$\tau_\theta = -\ \tau_{yx}$ cos 2_

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

To get the required equations for the case under consideration,let us add the respective equ ations for the above two cases such that

These are the equilibrium equation s for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour

This eqn gives two values of 2θ that differ by 180^0 .Hence the planes on which maximum and minimum normal stresses occurate 90^0 apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

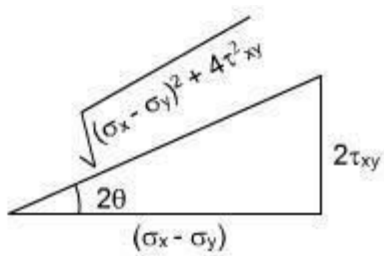
$$= 0$$

$$\text{i.e.} -(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$$

$$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$$

Thus, $$\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$$

From the triangle it may be determined



Substituting the values of cos2_ and sin2_ in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cdot\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{\tau_{xy}\cdot 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}\cdot(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

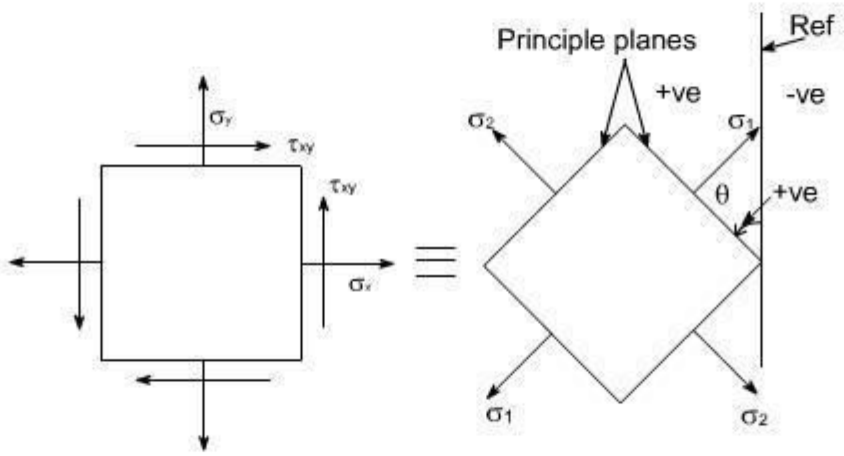
$$\tau_\theta = 0$$

This shows that the values oshear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero sheari ng stress. The maximum and minimum normal str esses are called the principal stresses, and the planes on which they act are called principal plane the soluti on of equation

will yield two values of 2θ separate d by 180^0 i.e. two values of θ separated by 90^0 .Thus the two principal stresses occur on mutually perpend icular planes termed principal planes.

Therefore the two 1 dimensional c omplex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a m aterial subjected to direct stresses the value of maximum shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$, Thus, for a 2-dimensional state of stress, subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Therefore,it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that an gles that locate the plane of maximum or minimum sheari g stresses form angles of 45^0 with the planes of principal stresses.

Futher, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

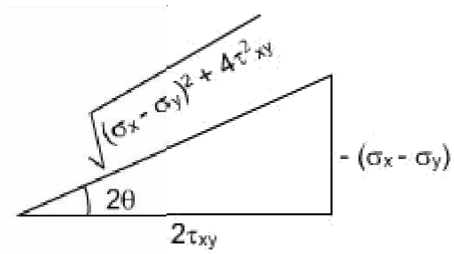
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}. - \frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2}.\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in si gn convention arises from the point of view of locating the planes on which shear stress act. From physi cal point of view these sign have no meaning.

The largest stress regard less of si gn is always know as maximum shear stress.

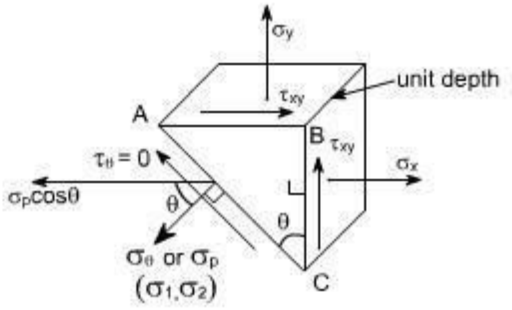
Principal plane inclination in ter ms of associated principal stress:

We know that the equation 

yields two values of q i.e. the inclin ation of the two principal planes on which the principal stre sses s$_1$ and s$_2$ act. It is uncertain,however, which stress acts on which plane unless equation.

 is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the a nswer to this problem in the following manner



Consider once again the equilibriu m of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses σ_p acts, and the shear stress is zero.

Resolving the forces horizontally w e get:

σ_x .BC . 1 + τ_{xy} .AB . 1 = σ_p . cosθ . AC dividing the above equation through by BC we get

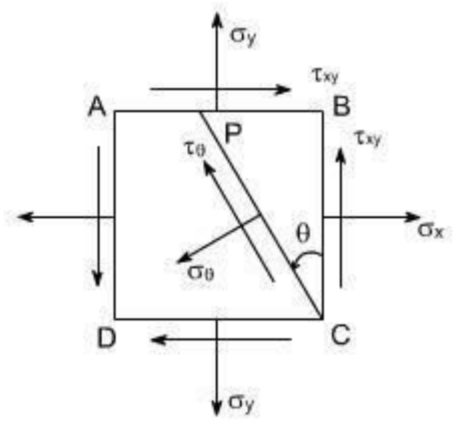
UNIV-IV Chapter 5

UNSYMMETRICAL BENDING AND SHEAR CENTRE

GRAPHICAL SOLUTION MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between nor mal and shear stresses acting on any inclined plan e at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a co mplete stress system for any condition of applied load in t wo dimensions

The Mohr's stress circle is used to find out graphically the direct stress σ and sheer stress_ on any plane inclined at θ to the plane on which $\sigma_x$acts.The direction of θ here is taken in anticlockwise dire ction from the BC.

STEPS:

In order to do achieve the desired o bjective we proceed in the following manner

(iv) Label the Block ABCD.

(v) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(vi) Plot the stresses on two adjace nt faces e.g. AB and BC, using the following sign convention.

Direct stresses_ tensile positive; compressive, negative

Shear stresses 1 tending to turn block clockwise, positive

1 tending to turn block counter clockwise, negative

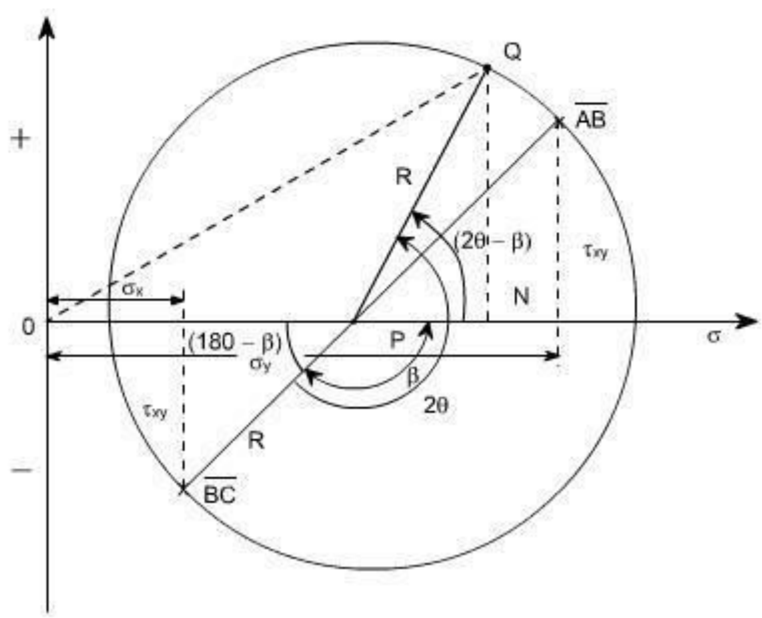
[i.e shearing stresses are +ve when its movement about the centre of the element is clockwise]

This gives two points on the graph which may than be labeled as [illegible] respectively to denote stresses on these planes.

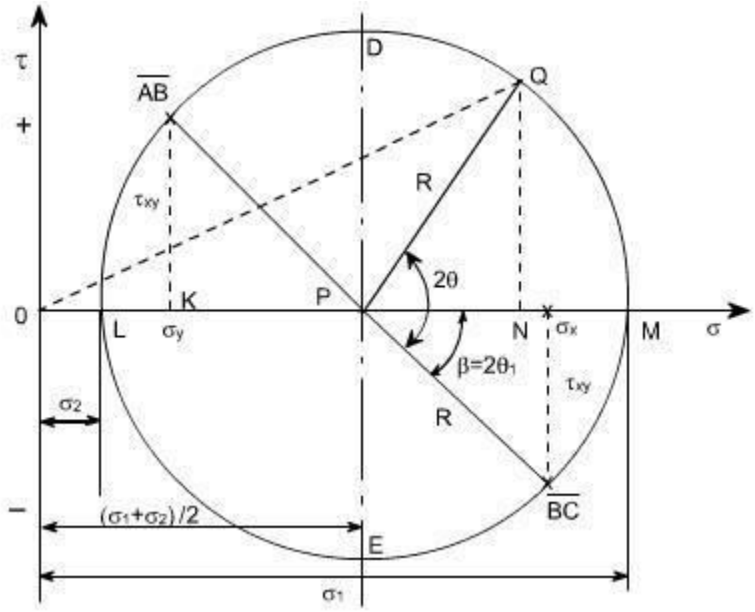
- Join [illegible].

- The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining [illegible] is diameter. Therefore the circle can now be drawn.

Now every point on the circle then r epresents a state of stress on some plane through C.



Proof:



Consider any point Q on the circum ference of the circle, such that PQ makes an angle 2 _with BC, and drop a perpendicular from Q to meet the s axis at N.Then OQ represents the resultant stress on the plane an angle θ to BC. Here we have assu med that $\sigma_x > \sigma_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

$$ON = OP + PN$$

$$OP = OK + KP$$

$$OP = \sigma_y + 1/2\,(\sigma_x - \sigma_y)$$

$$= \sigma_y / 2 + \sigma_y / 2 + \sigma_x / 2 + \sigma_y / 2$$

$$= (\sigma_x + \sigma_y) / 2$$

$PN = R\cos(2\theta - \beta)$

hence $ON = OP + PN$

$$= (\sigma_x + \sigma_y) / 2 + R\cos(2\theta - \_)$$

$$= (\_{}_x + \sigma_y) / 2 + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

now make the substitutions for $R\cos\beta$ and $R\sin\beta$.

Thus,

$$ON = 1/2\,(\sigma_x + \sigma_y) + 1/2\,(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

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$= R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$

Thus, substituting the values of $R\cos\beta$ and $R\sin\beta$, we get

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If we examine the equation (1) and (2), we see that this is the same equation which we have already derived analytically

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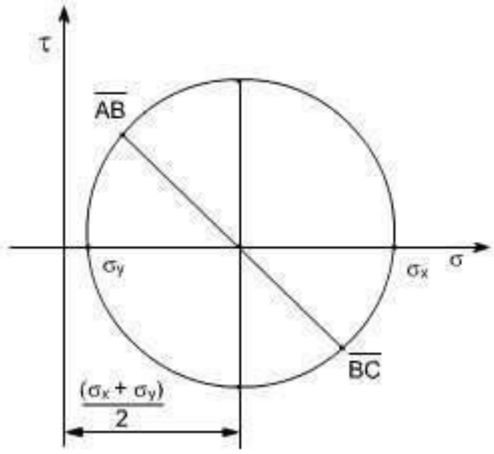

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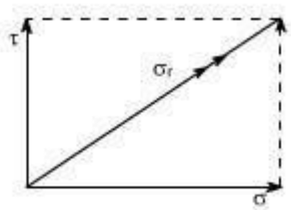

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Now the normal stress on an obliqe plane is given by the relation

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$\theta = 50^0 68'$

The shear stress on the oblique plane is then given by

$\tau\_ = 1/2\ \sigma_y \sin 2\theta$

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For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

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(b) $25\ MN/m^2$ tensile at right angles to (a)

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Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

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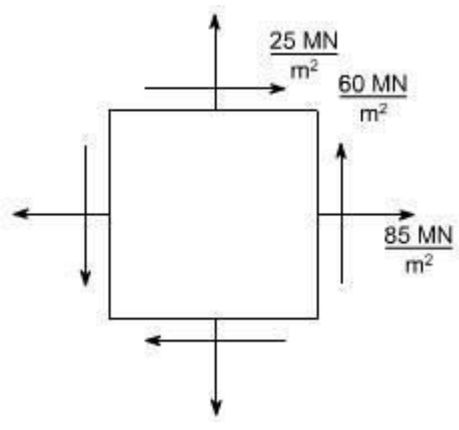

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σ_1 and σ_2

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$$= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4 \times 60^2)}$$

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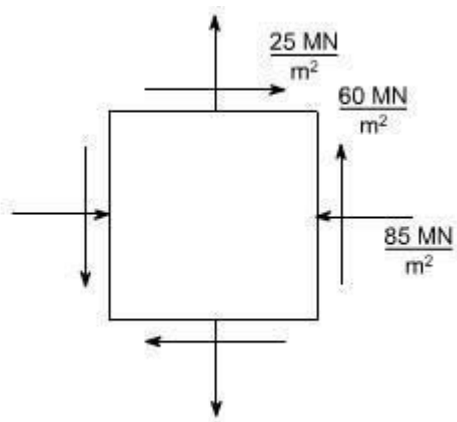

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$$\sigma_1 \,\&\, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4 \text{ MN/m}^2;\ \sigma_2 = -111.4 \text{ MN/m}^2$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

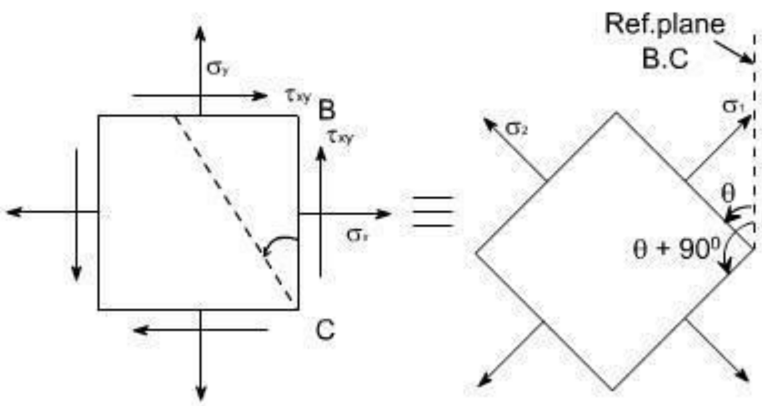
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

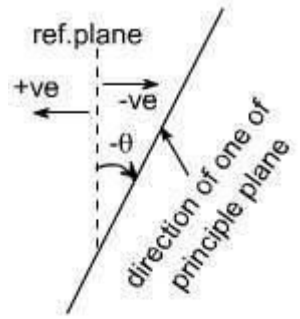
$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \ \theta = -23.74^0$$

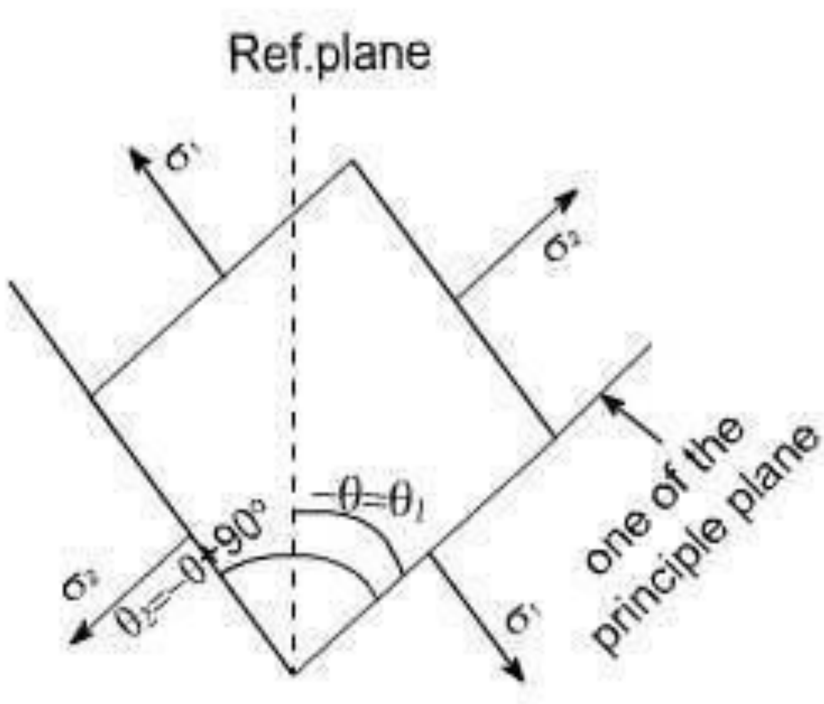
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



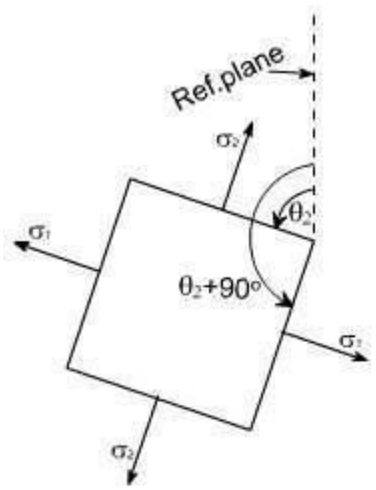
So this is the direction of one principle plane & the principle stresses acting on this would be σ_1 when is acting normal to this plane, now the direction of other principal plane would be $90^0 + \theta$ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane $\theta + 90^0$ in the same direction to get the another plane, n ow complete the material element if θ is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be $\{-\theta + 90\}$ since the angle $-\theta$ is always less in magnitude then 90 hence the quantity $(-\theta + 90)$ would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, than rotate this block through 180^0 so as to have the following appearance.



So whenever one of the angles comes negative to get the positive value,

first Add 90^0 to the value and again add 90^0 as in this case $\theta = -23^0 74'$

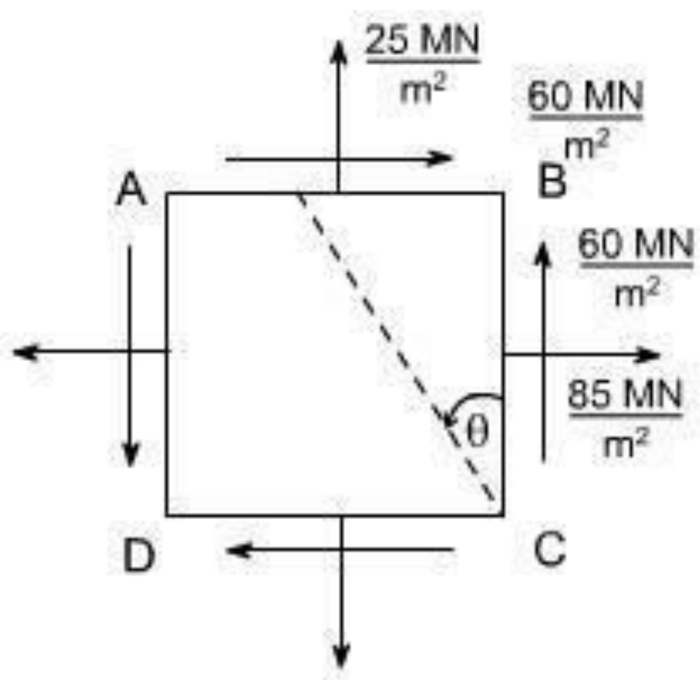
so $\theta_1 = -23^0 74' + 90^0 = 66^0 26'$ .Again adding 90^0 also gives the direction of other principle planes

i.e $\theta_2 = 66^0 26' + 90^0 = 156^0 26'$

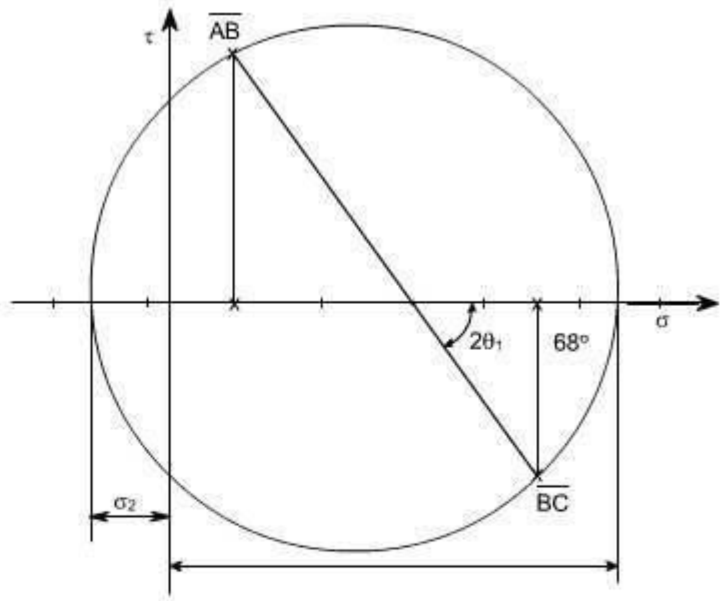
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e the Mohr's stress circle,for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

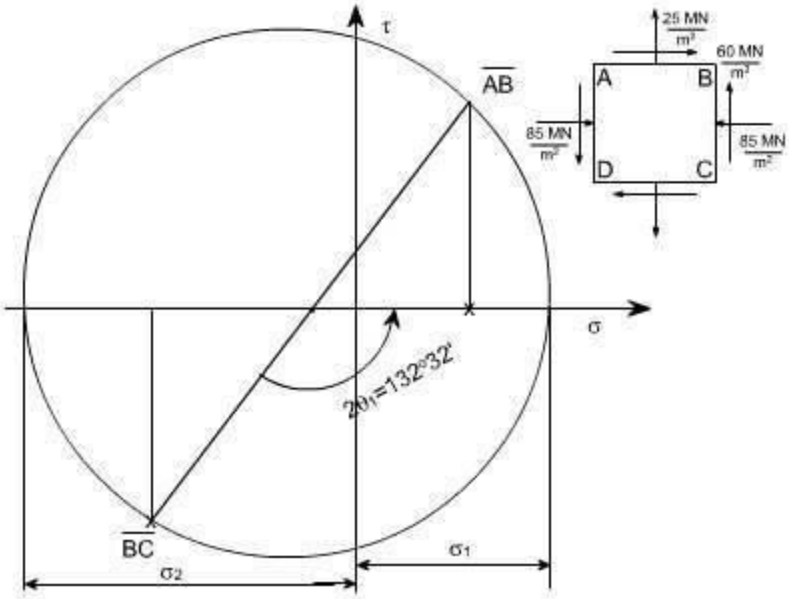
σ_1 = 120 MN/m^2 tensile

σ_2 = 10 MN/m^2 compressive

θ_1 = 34^0 counter clockwise from BC

θ_2 = 34^0 + 90 = 124^0 counter clockwise from BC

Part Second : The required configuration i.e the block diagram for this case is shown along with the stress circle.



By taking the measurements, the various quantites computed are given as

$\sigma_1 = 56.5\ MN/m^2$ tensile

$\sigma_2 = 106\ MN/m^2$ compressive

$\theta_1 = 66^0\ 15'$ counter clockwise from BC

$\theta_2 = 156^0\ 15'$ counter clockwise from BC

Salient points of Mohr's stress circle:

1. complementary shear stresses (on planes 90^0 apart on the circle) are equal in magnitude

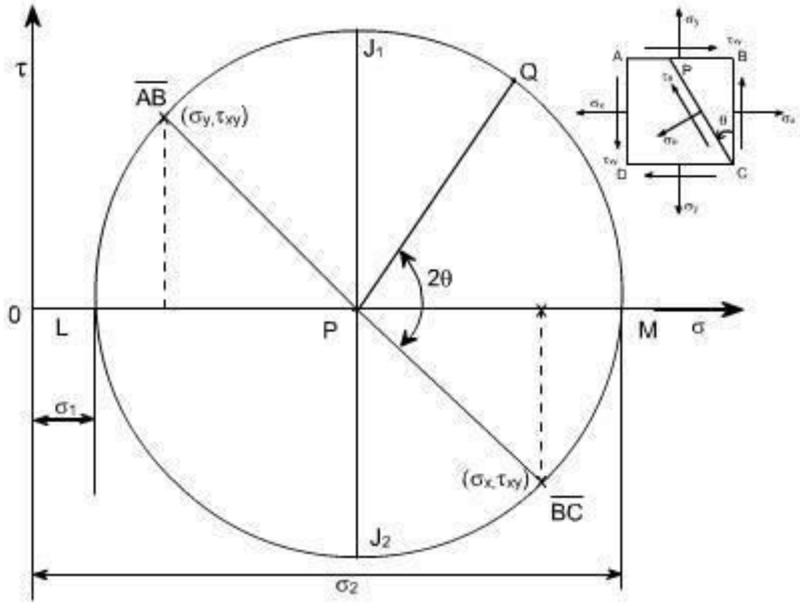
2. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^0 apart in material)

3. There are no shear stresses on principal planes: point L and M lie on normal stress axis.

4. The planes of maximum shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.

5. The maximum shear stresses are equal in magnitude and given by points D and E

6. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represen ts all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point 1Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress com ponents on any plane passing through the point ca n be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the elem ent ABCD, which are 90^0 apart, are represented on the circle by [redacted] and they are 180^0 apart.

2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can b e seen at a point. Thus, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and therefore 90^0 apart in the material, on which shear stress τ_θ is zero. These planes are termed as principal planes a nd normal stresses acting on them are known as principal stresses.

Thus , σ_1 = OL

σ_2 = OM

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radius of i.e. τ_{max}= 1/2(_ 1− σ_2),the corresponding normal stress is obviously the distance OP = 1/2 (_ x+ σ_y) , Further it can also be seen that the planes on which the shear stres s is maximum are situated 90^0 from the principal planes (on circle), and

45^0 in the material.

4. The minimum normal stress is jus t as important as the maximum. The algebraic minimum s tress could have a magnitude greater than that of the maximum principal stress if the state of stress wer e such that the centre of the circle is to the left of orgin.

i.e. if $\sigma_1 = 20$ MN/m^2 (say)

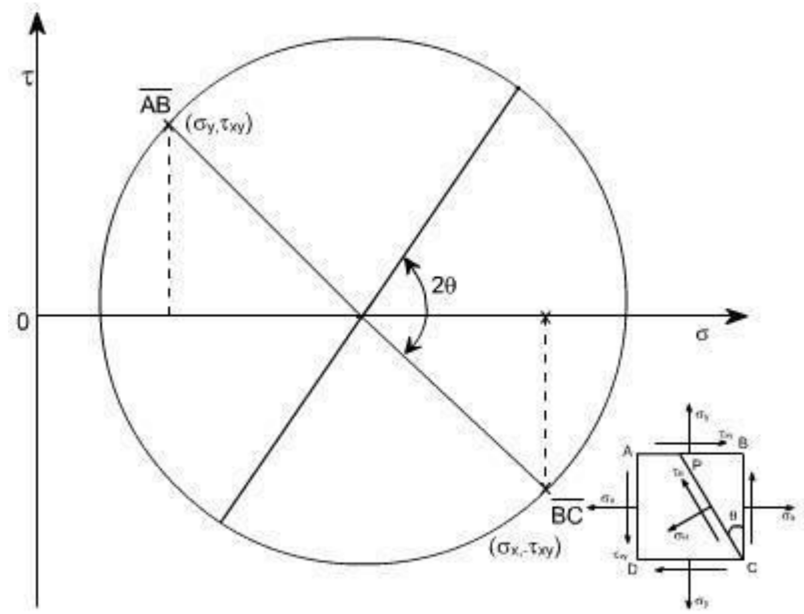
$\sigma_2 = -80$ MN/m^2 (say)

Then $\tau_{max}{}^m = (\sigma_1 - \sigma_2 / 2) = 50$ MN/m^2

If should be noted that the principal stresses are considered a maximum or minimum mathem atically e.g. a compressive or negative stress is l ess than a positive stress, irrespective or numerical value.

5. Since the stresses on perpendul ar faces of any element are given by the co-ordinates of t wo diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. Thus sum is an invariant for any particular state of stress.

Sum of the two normal stress comp onents acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the o rientation of these planes.



This can be also understand from t he circle Since AB and BC are diametrically opposite thus, what ever may be their orientation, they will always lie on the diametre or we can say that their sum won't change, it can also be seen from analytical relatio ns

We know 

on plane BC; θ = 0

$\sigma_{n1} = \sigma_x$

on plane AB; θ = 270^0

$\sigma_{n2} = \sigma_y$

Thus $\sigma_{n1} + \sigma_{n2} = \sigma_x + \sigma_y$

6. If $\sigma_1 = \sigma_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are d eveloped on xy plane.

7. If $\sigma_x + \sigma_y = 0$, then the center of Mohr's circle coincides with the origin of σ − τ co-ordinates.

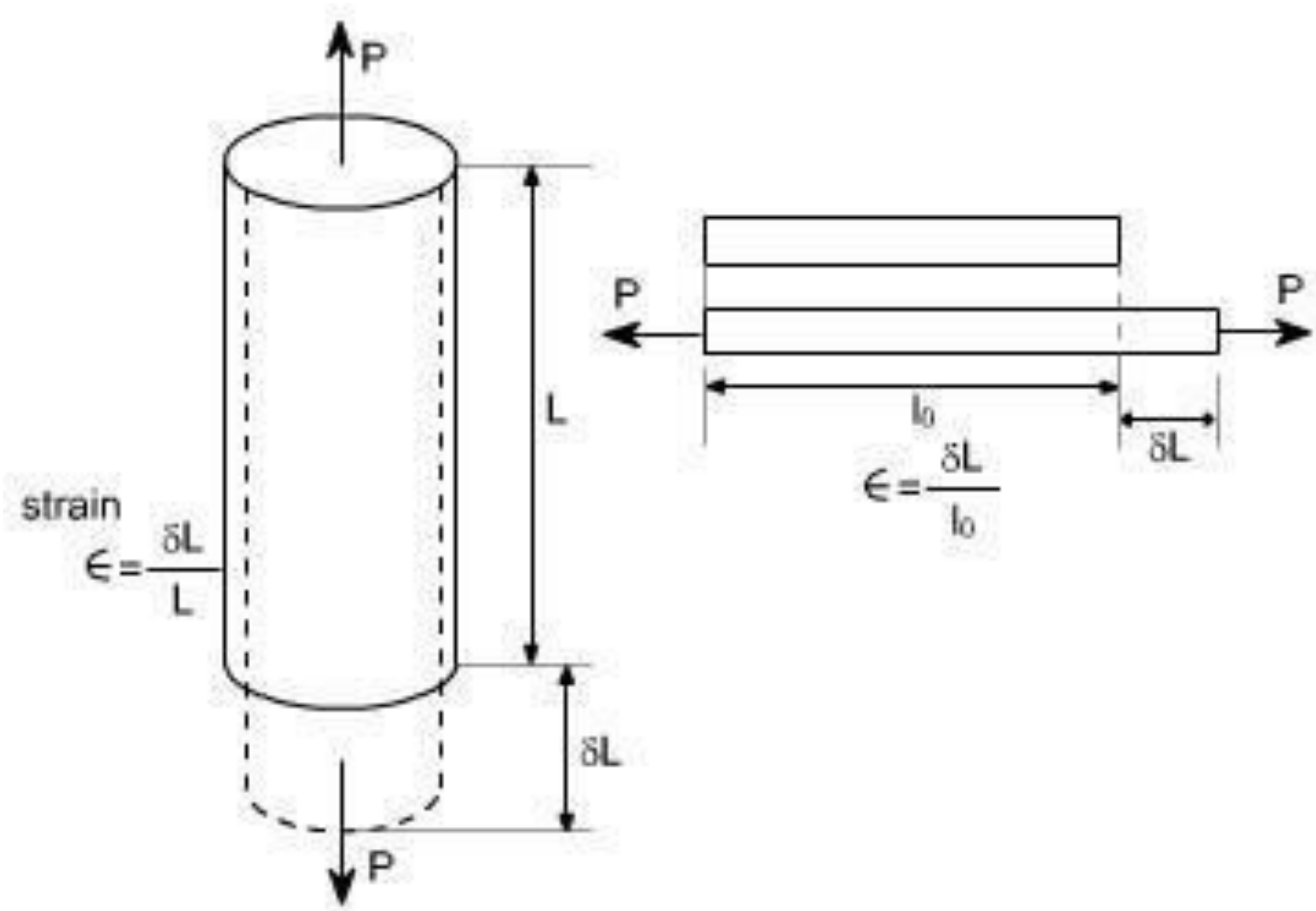
ANALYSIS OF STRAINS

CONCEPT OF STRAIN

Concept of strain : if a bar is subj ected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount δL, the strain produce is defined as follows:

[illegible]

Strain is thus, a measure of the deformation of the material and is a nondimensional Quantity i.e. it has no units. It is simply a ratio of two qua ntities with the same unit.



Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becom es $\propto \in$.

Sign convention for strain:

Tensile strains are positive wherea s compressive strains are negative. The strain defined earlier was known as linear strain or normal strain or the longitudinal strain now let us define the shear strain.

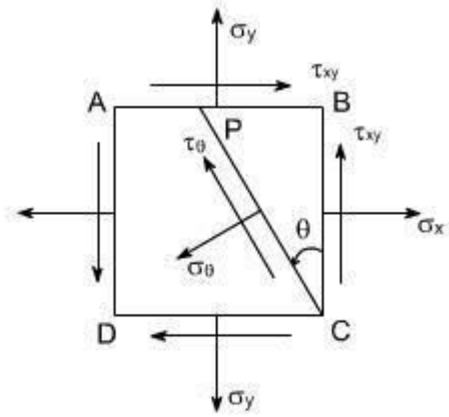
Chapter 5 Chapter 5

UNSYMMETRICAL BENDING AND SHEAR CENTRE

GRAPHICAL SOLUTION MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between nor mal and shear stresses acting on any inclined plan e at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a co mplete stress system for any condition of applied load in t wo dimensions

The Mohr's stress circle is used to find out graphically the direct stress σ and sheer stress_ on any plane inclined at θ to the plane on which $\sigma_x$acts.The direction of θ here is taken in anticlockwise dire ction from the BC.

STEPS:

In order to do achieve the desired o bjective we proceed in the following manner

(vii) Label the Block ABCD.

(viii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(ix) Plot the stresses on two adjace nt faces e.g. AB and BC, using the following sign convention.

Direct stresses_ tensile positive; compressive, negative

Shear stresses 1 tending to turn block clockwise, positive

1 tending to turn block counter clockwise, negative

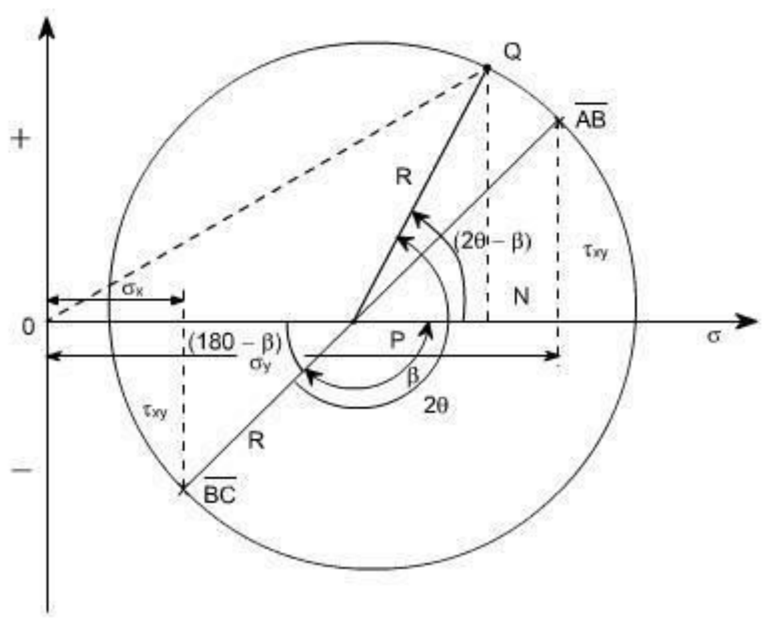
[i.e shearing stresses are +ve when its movement about the centre of the element is clockwise]

This gives two points on the graph which may than be labeled as [illegible] respectively to denote stresses on these planes.

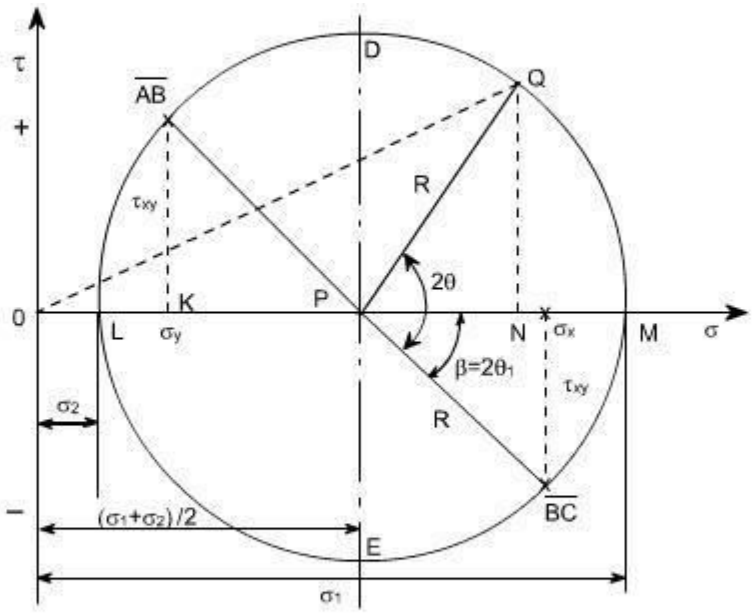
- Join [illegible].

- The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining [illegible] is diameter. Therefore the circle can now be drawn.

Now every point on the circle then r epresents a state of stress on some plane through C.



Proof:



Consider any point Q on the circum ference of the circle, such that PQ makes an angle 2 _with BC, and drop a perpendicular from Q to meet the s axis at N.Then OQ represents the resultant stress on the plane an angle θ to BC. Here we have assu med that $\sigma_x > \sigma_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

$$ON = OP + PN$$

$$OP = OK + KP$$

$$OP = \sigma_y + 1/2\ (\sigma_x - \sigma_y)$$

$$= \sigma_y / 2 + \sigma_y / 2 + \sigma_x / 2 + \sigma_y / 2$$

$$= (\sigma_x + \sigma_y) / 2$$

$PN = R\cos(2\theta - \beta)$

hence ON = OP + PN

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$$= (\_{}_x + \sigma_y) / 2 + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

now make the substitutions for $R\cos\beta$ and $R\sin\beta$.

[redacted]

Thus,

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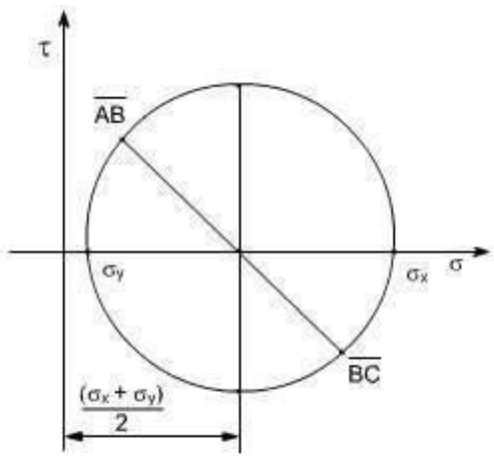

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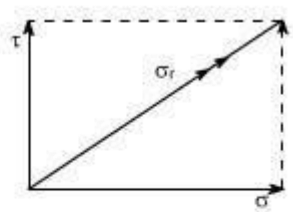

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The shear stress on the oblique plane is then given by

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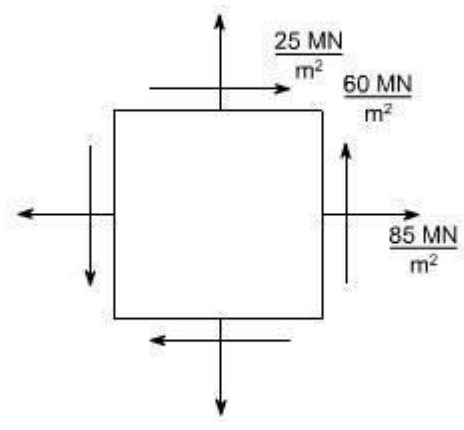

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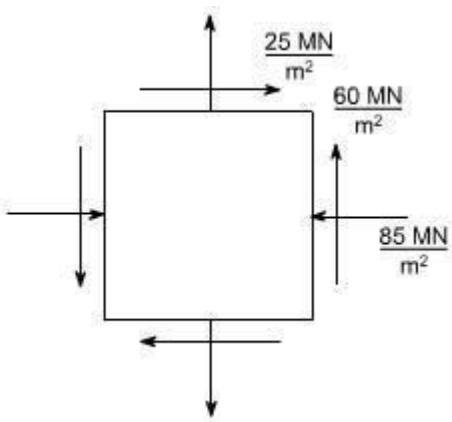
The solution of this equation will yeild two values θ i.e they θ_1 and θ_2 giving $\theta_1 = 31^0 71'$ & $\theta_2 = 121^0 71'$

(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.



Again the principal stresses would be given by the equation.

$$\sigma_1 \& \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4 \text{ MN/m}^2;\ \sigma_2 = -111.4 \text{ MN/m}^2$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

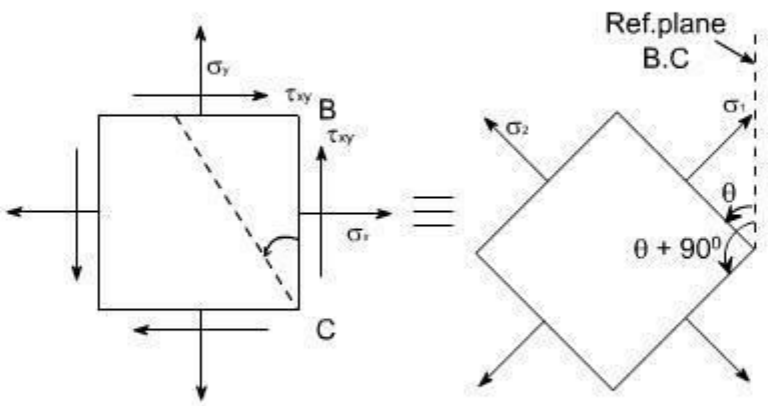
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

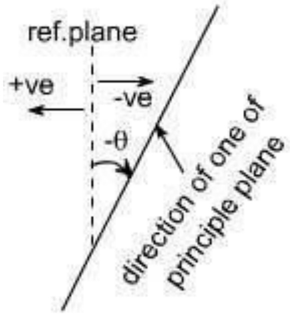
$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \theta = -23.74^0$$

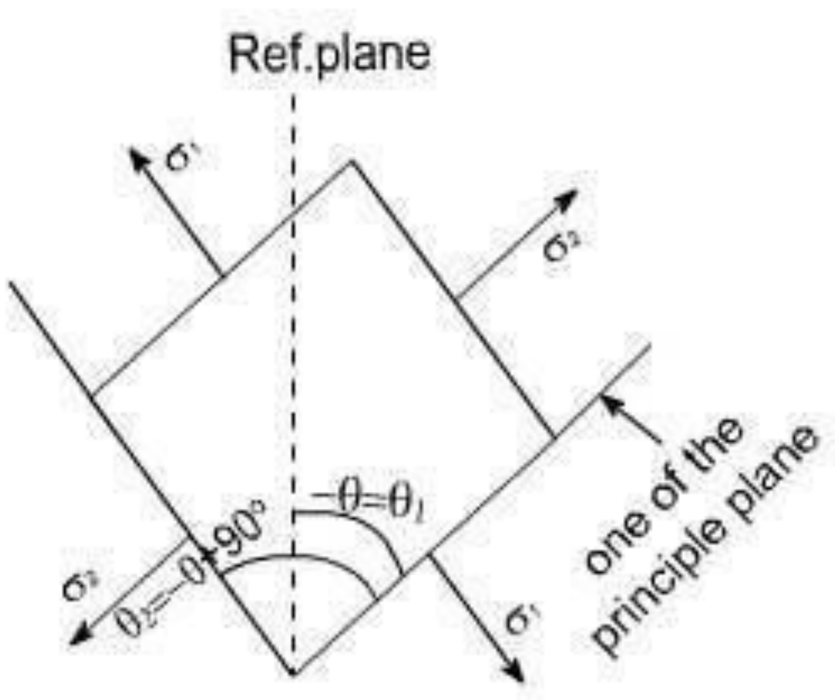
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



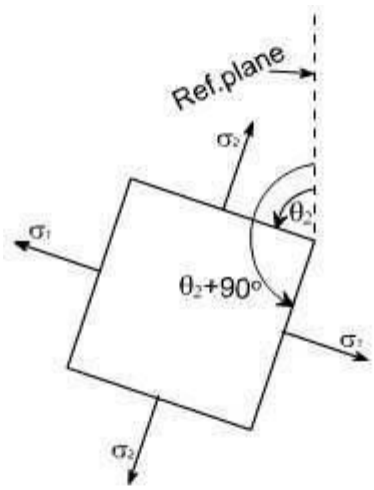
So this is the direction of one principle plane & the principle stresses acting on this would be σ_1 when is acting normal to this plane, now the direction of other principal plane would be $90^0 + \theta$ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane $\theta + 90^0$ in the same direction to get the another plane, n ow complete the material element if θ is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be {$-\theta$ + 90} since the angle $-\theta$ is always less in magnitude then 90 hence the quantity ($-\theta$ + 90) would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, than rotate this block through 180^0 so as to have the following appearance.



So whenever one of the angles comes negative to get the positive value,

first Add 90^0 to the value and again add 90^0 as in this case $\theta = -23^0 74'$

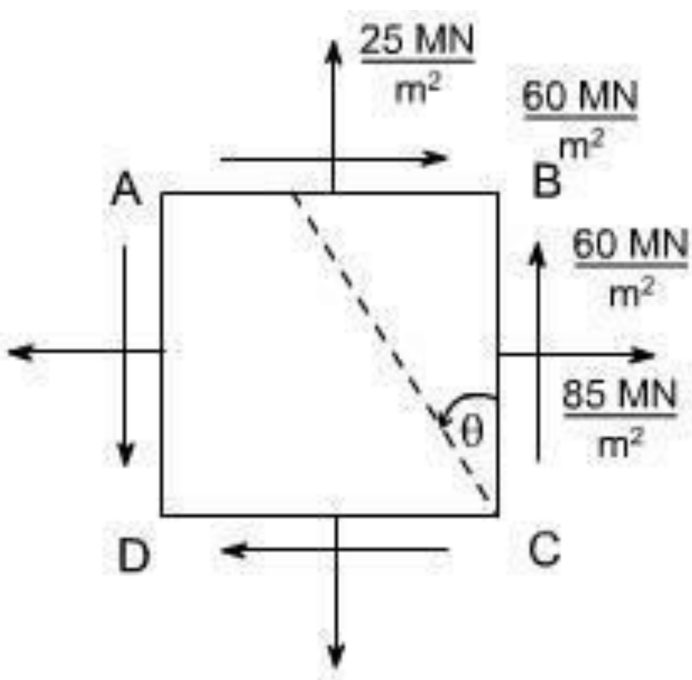
so $\theta_1 = -23^0 74' + 90^0 = 66^0 26'$ .Again adding 90^0 also gives the direction of other principle planes

i.e $\theta_2 = 66^0 26' + 90^0 = 156^0 26'$

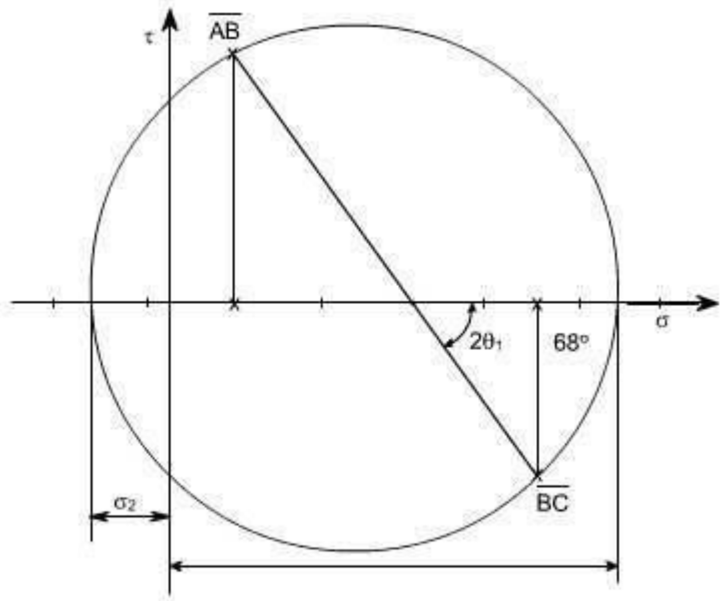
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e the Mohr's stress circle,for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

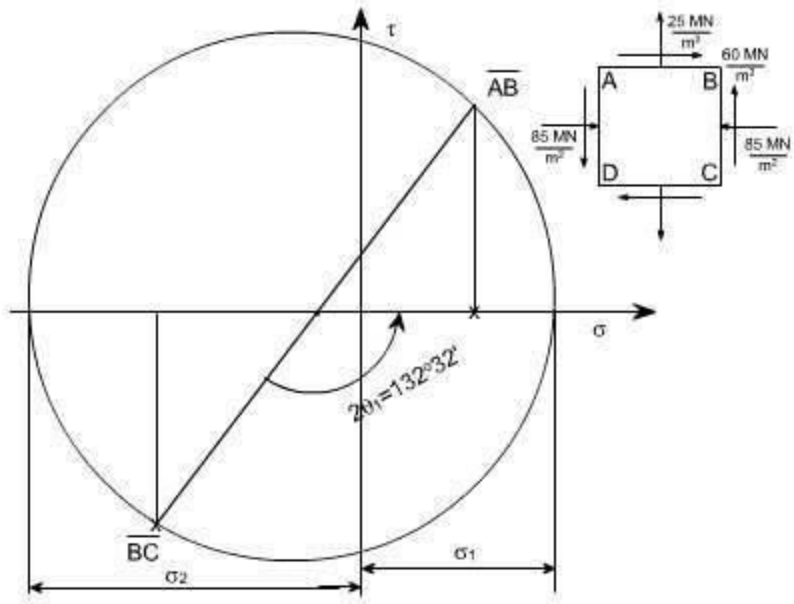
$\sigma_1 = 120$ MN/m^2 tensile

$\sigma_2 = 10$ MN/m^2 compressive

$\theta_1 = 34^0$ counter clockwise from BC

$\theta_2 = 34^0 + 90 = 124^0$ counter clockwise from BC

Part Second : The required configuration i.e the block diagram for this case is shown along with the stress circle.



By taking the measurements, the various quantites computed are given as

σ_1 = 56.5 MN/m^2 tensile

σ_2 = 106 MN/m^2 compressive

θ_1 = 66^0 15' counter clockwise from BC

θ_2 = 156^0 15' counter clockwise from BC

Salient points of Mohr's stress circle:

7. complementary shear stresses (on planes 90^0 apart on the circle) are equal in magnitude

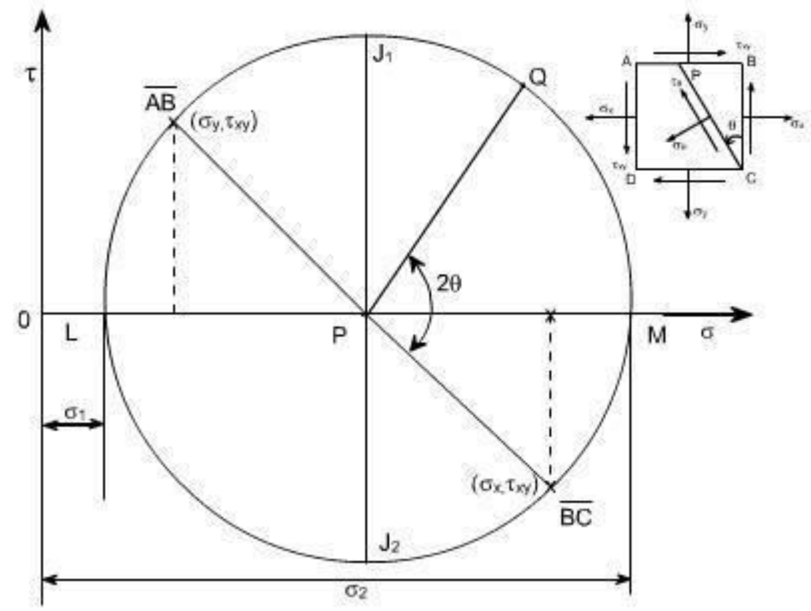
8. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^0 apart in material)

9. There are no shear stresses on principal planes: point L and M lie on normal stress axis.

10. The planes of maximum shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.

11. The maximum shear stresses are equal in magnitude and given by points D and E

12. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represen ts all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point 1Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress com ponents on any plane passing through the point ca n be found using Mohr's circle. Worthy of note:

8. The sides AB and BC of the elem ent ABCD, which are 90^0 apart, are represented on the circle by [redacted] and they are 180^0 apart.

9. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can b e seen at a point. Thus, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and therefore 90^0 apart in the material, on which shear stress τ_θ is zero. These planes are termed as principal planes a nd normal stresses acting on them are known as principal stresses.

Thus , σ_1 = OL

σ_2 = OM

10. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radius of i.e. τ_{max}= 1/2(_ 1$-$ σ_2),the corresponding normal stress is obviously the distance OP = 1/2 (_ x+ σ_y) , Further it can also be seen that the planes on which the shear stres s is maximum are situated 90^0 from the principal planes (on circle), and

45^0 in the material.

11. The minimum normal stress is jus t as important as the maximum. The algebraic minimum s tress could have a magnitude greater than that of the maximum principal stress if the state of stress wer e such that the centre of the circle is to the left of orgin.

i.e. if σ_1 = 20 MN/m^2 (say)

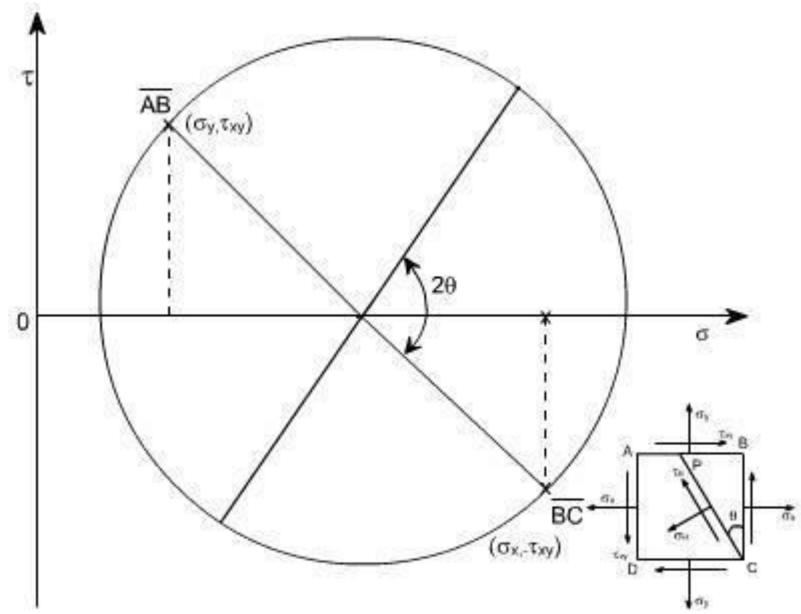
σ_2 = $-$80 MN/m^2 (say)

Then $\tau_{max}{}^m$ = ($\sigma_1 - \sigma_2$ / 2) = 50 MN/m^2

If should be noted that the principal stresses are considered a maximum or minimum mathem atically e.g. a compressive or negative stress is l ess than a positive stress, irrespective or numerical value.

12. Since the stresses on perpendul ar faces of any element are given by the co-ordinates of t wo diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. Thus sum is an invariant for any particular state of stress.

Sum of the two normal stress comp onents acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the o rientation of these planes.



This can be also understand from t he circle Since AB and BC are diametrically opposite thus, what ever may be their orientation, they will always lie on the diametre or we can say that their sum won't change, it can also be seen from analytical relatio ns

We know 

on plane BC; $\theta = 0$

$\sigma_{n1} = \sigma_x$

on plane AB; $\theta = 270^0$

$\sigma_{n2} = \sigma_y$

Thus $\sigma_{n1} + \sigma_{n2} = \sigma_x + \sigma_y$

13. If $\sigma_1 = \sigma_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are d eveloped on xy plane.

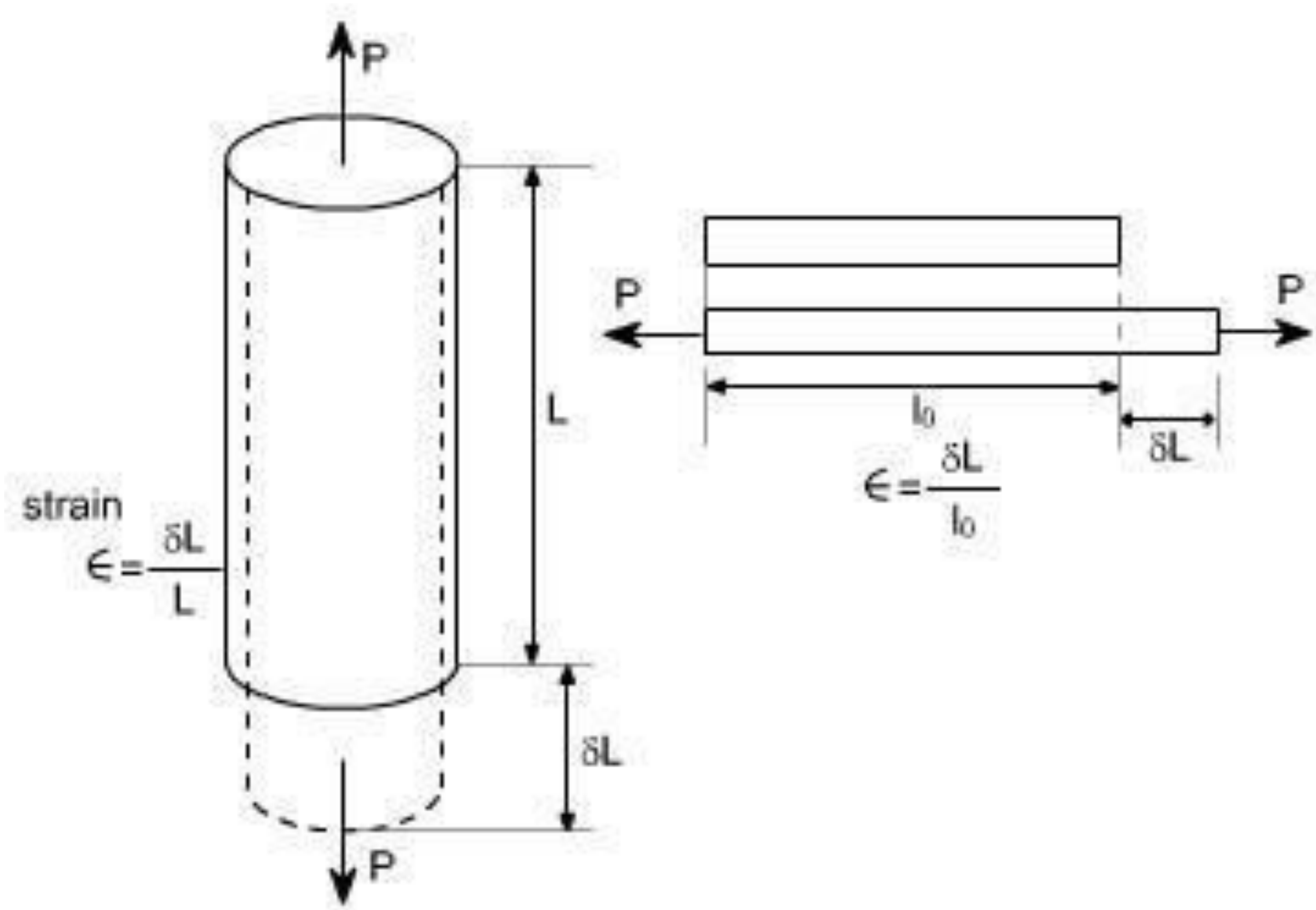
14. If $\sigma_x + \sigma_y = 0$, then the center of Mohr's circle coincides with the origin of $\sigma - \tau$ co-ordinates.

ANALYSIS OF STRAINS

CONCEPT OF STRAIN

Concept of strain : if a bar is subj ected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount δL, the strain produce is defined as follows:

Strain is thus, a measure of the deformation of the material and is a nondimensional Quantity i.e. it has no units. It is simply a ratio of two qua ntities with the same unit.



Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becom es $\propto \in$.

Sign convention for strain:

Tensile strains are positive wherea s compressive strains are negative. The strain defined earlier was known as linear strain or normal strain or the longitudinal strain now let us define the shear strain.