UNIT –I
(FUNCTIONS COMPLEX VARIABLES)

- Defn: A number of the form x+iy , where x and y are real numbers and i= $\sqrt{(-1)}$ is called a complex number.x is jcalled the real part of x+iy and y is called the imaginary part written R(x+iy), I(x+iy) respectively.
- Properties:
    1) If x+iy=u+iv then x-iy=u-iv
    2) Two complex numbers x+iy and u+iv are said to be equal where R(x+iy)=R(u+iv) is x=u, I(x+iy)=I(u+iv) i.e y=v
    3) Sum, difference, product and quotient of any complex numbers is itself a csomplex number.
    4) Every complex number x+iy can always be expressed in the form $r\,(\cos\theta + i\sin\theta)$
- Defn: The number $r = +\sqrt{x^2+y^2}$ is called the modulus of x+iy and is written as mod(x+iy) or (x+iy) the angle θ is called the amplitude of argument of x+iy and is written as amp(x+iy) or arg(x+iy).

Evidently, the amplitude θ has an infinite number of values. The value of θ which lie

between $-\Pi$ and Π is called the principal value of the Amplitude.

- If the conjugate of Z = x+iy be $\overline{Z}$ then R(Z) = $\frac{1}{2}(Z+\overline{Z})$ and I(Z) = $\frac{1}{2i}(Z-\overline{Z})$

$|Z| = \sqrt{R^2(Z)+I^2(Z)} = |\overline{Z}|$

$|Z|^2 = Z\overline{Z}$

$\overline{Z_1+Z_2} = \overline{Z}_1+\overline{Z}_2$

$\overline{Z_1\cdot Z_2} = \overline{Z}_1\cdot\overline{Z}_2$

$\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z}_1}{\overline{Z}_2}$ **where** $\overline{Z}_2 \neq 0$

- The point whose Cartesian coordinates are (x,y) uniquely represents the complex number z = x+iy on the complex plane Z. The diagram in which the representation is carried out is called the argand's diagram.
- If Z_1, Z_2 are two complex numbers then

1. $|Z_1+Z_2| \le |Z_1|+|Z_2|$

2. $|Z_1-Z_2| \ge |Z_1|+|Z_2|$

In general $|Z_1+Z_2+......+Z_n| \le |Z_1| + |Z_2| +......+ |Z_n|$

3. amp (Z_1Z_2) = amp (Z_1) + amp (Z_2)
4. $|Z_1/Z_2|=|Z_1| / |Z_2|$
5. amp (Z_1 / Z_2) = amp (Z_1)- amp(Z_2)

- Demoivre's theorem: If n be I) an integer positive or negative then
  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

- If x and y are real variables then z = x+iy is called complex variable. If corresponding to each value of the complex variable z(= x+iy) in a region R there corresponds one or more values of another complex variable w (= u+iv) then w is called a function of complex variable z and is denoted by w =f(z) = u+iv where u,v are real and imaginary parts of w and the function of real variables
  w =f(z) =u(x,y)+iv(x,y)

- If to each value of z there corresponds one and only one value of w then w is called a single valued function of z.
- If to each value of z there corresponds more than one value of w then w is called multi valued function of z

- To represent w =f(z) graphically, we take two argand diagrams one to represent the point z and the other to represent the point w
- The distance between the point z and 'a' is denoted by |z-a|
- A circle of radius 'd' with center at 'a' is denoted by |z-a| = d.
- The inequality |z-a|< d denoted by every point inside the circle C: |z-a|<d i.e., it represents the interior of the circle excluding its circumference. The interior of the circle including its circumference is denoted by $|z-a| \leq d$.
- The Neighborhood of a point 'a' is represented by the inequality |z-a|<d.
- |z-a| >d represents the exterior of the circle with center at 'a' and radius 'd'.
- The region between two concentric circles of radii d_1 and d_2 ($d_1 > d_2$) can be represented by $d_1 < |z-a| < d_2$
- The equation |z| =1 represents a unit circle about origin.
- If there exists a circle with center at origin enclosing all points of a region R then R is said to be bounded.
- If a region is defined to include all the points on its various boundary curves, it is said to be closed.
- If R contains none of its boundary points, it is said to be open.
- A set of points in the complex plane S is called open if every point of S has a ngd. All the points of which belong to S.
- A set of points in the complex plane S is called closed if the points which do not belong to S form an open set S.

Limit of f(z): A function w= f(z) tends to the limit 'ℓ' as z approaches a point z_0 along any path, if to ' each positive arbitrary number ε, however small there corresponds a positive number δ, such that $|f(z)- \ell| < \varepsilon$ whenever $0<|z- z_0|<\delta$

i.e., $\ell - \varepsilon < f(z) < \ell + \varepsilon$ whenever $z_0-\delta < z < z_0+\delta$, $z \neq z_0$

We write $\lim\limits_{z\to z_0} = \ell$

- In real variables $x \to x_0$ implies x approaches x_0 along the line either from left or right.

- In complex variables $z \to z_0$ implies z approaches z_0 along the path (straight or curved) since a complex plane can be joined by infinite number of curves.

Continuity of f(z): A single valued function f(z) is said to be continuous at a point $z = z_0$ if $\lim_{z \to z_0} = f(z_0)$

- A function f(z) is said to be continuous in a region R in the Z-plane if it is continuous at every point of the region.
- If w= f(z)= U(x,y)+iv(x,y) is continuous at $z= z_0$ then u(x,y) and v(x,y) are also continuous at $z= z_0$ i.e., at $x= x_0$ and $y=y_0$.

Conversely, if u(x,y) and v(x,y) are continuous at (x_0,y_0) then f(z) will be continuous at $z=z_0$

- Sum, difference and product of two continuous functions is continuous .Quotient function of two continuous if exists then it is also continuous .If f(z) is continuous |f(z)| is also continuous.

Differentiablity:

A single valued function f(z) is differentiable at the point $z=z_0$ is denoted by f '(z) or $\frac{dw}{dz}$ and is defined by the equation $f^1(z_o)= \lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z}$ provided the limit exists.

Analytic function: A single valued function f(z) is said to be analytic at a point z_0 if it has a unique derivative at z_0 and at every point in the neighborhood of z_0

Cauchy-Riemann Equations:

<u>**Cartesian form:**</u> The necessary and sufficient condition for for function w=f(z) to be analytic in a Region R are a) The four first order derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists and are continuous in R. b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

- The conditions given in b) are called cauchy-Riemann equation or C.R. Equations.

Polar form: Let (r,θ) be the polar co-ordinates of the point whose Cartesian co-ordinates are (x,y) with $x = r\cos\theta$ $y = r\sin\theta$.The C.R. Equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Harmonic Function: Any function $\phi(x,y)$ which possess continuous partial derivatives of the first and second orders and satisfy Laplace equations $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Harmonic function.

Conjugate Harmonic function: If a function u(x,y) is Harmonic in the domain and if we find another Harmonic function v(x,y) such that they satisfy the cauchy- Riemann equations and Laplace equations then we say v(x,y) is harmonic conjugate of u(x,y).

Properties of Analytic Functions:

- An analytic function with constant real part is constant.
- An analytic function with constant imaginary part is constant.
- An analytic function with constant modulus is constant.
- The real and imaginary parts of an analytic functions are harmonic
- Every analytic function f(z) = u+iv defines two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ forms an orthogonal system.
- An analytic function can be easily constructed by using Milne –Thomson method.

Elementary functions:

$\sin\theta = (e^{i\theta} - e^{-i\theta})/2$ $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$

Sin ix = i sinh x cosix = coshx

Sinh ix = isinx coshix = cosx

Tanix =itanhx Tanhix = i tanx

Complex potential function: The analytic function $w = \phi(x,y)+i\psi(x,y)$ is called complex potential function. Its real part $\phi(x,y)$ represents the velocity potential function and its imaginary part $\psi(x,y)$ represents the stream function.

Both ϕ,ψ satisfy Laplace equation. Given any one of them we find the other.

Essay Questions:

1. Separate the real and imaginary parts of a) tan(x+iy) b) sec(x+iy)

2. Find the general values of log(1+i)

3. Find all the roots of sinz =2

4. Find the values of i^i and $\log(i^i)$

5. State and prove the necessary and sufficient condition for analyticity.

6. Show that both real and imaginary parts of an analytic function are harmonic
7. Prove that every analytic function f(z) = u+iv defines two families of curves $u(x,y)=c_1$ and $v(x,y) = c_2$ forming an orthogonal system.
8. Define Cauchy-Reimann equations in polar form.
9. Prove that the function f(z) defined by $f(z) = \dfrac{x^3(i+1)-y^3(1-i)}{x^2+y^2}$ $(z \neq 0)$, f(0) = 0 is continuous and the cauchy-Riemann equations are satisfied at origin.
10. Show that $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfy laplace equations. Find the corresponding analytical function.
11. Find the analytic function whose real part is $u = e^x[(x^2-y^2)\cos y - 2xy\sin y]$
12. If $w = \phi+i\psi$ represents the complex potential transform electric field and

$\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$. Determine the function ϕ

13. If $w = \log z$. Find dw/dz and determine where w is not analytic.
14. Find the conjugate harmonic function of the harmonic function $u = x^2 - y^2$
15. If $f(z) = u+iv$ is analytic the prove that a) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ and

b) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)U^p = p(p-1)U^{p-2}|f'(z)|^2$

16. If $f(z) = u+iv$ is an analytic function of z and if $u-v = e^x(\cos y - \sin y)$ find f(z) in terms of z.
17. S.T the function $f(z) = |z|^2$ is differentiable only at the origin
18. S.T the following functions are harmonic and also find the conjugate harmonic function

    i) $u = \dfrac{1}{2}\log(x^2+y^2)$ ii) $u = 4xy-3x+2$ iii) $u = e^{2x}(x\cos 2y - y\sin 2y)$

19. .Find the analytic function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$
20. If f(z) is analytic function with constant modulus s.t f(z) is constant
21. If the potential function is $\log(x^2+y^2)$ find the flux function and the complex potential function
22. S.T $u = e^{-2xy}\sin(x^2-y^2)$ is harmonic find the conjugate function v and express u+iv as an analytic function of z
23. Determine whether the function sinxsiny-icosxcosy is analytic function of complex variable z = x+iy.
24. S.T $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$ $z \neq 0$ & $f(0) = 0$ is analytic at z = 0
25. S.T $f(z) = xy + iy$ is continuous every where but it is not analytic any where
26. Find the analytic function $w = u+iv$ if $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$
27. If $f(z) = u + iv$ is analytic function and $u-v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ find f(z) subject to the condition $f(\pi\backslash 2)$.

UNIT-II
(COMPLEX INTEGRATION)

Complex Line integral: Let f(z) be a function which is continous at all points on the curve C whose end points are A,B

Dividing the curve C into n parts by the points z_0 (= A),z_1,z_2,.....z_n (=B). Let f(z) be defined at all these points.Let z_r be a point on the arc joining z_{r-1} to z_r. Let z_r-z_{r-1}=δ z_r. Define the sum S_n = $\sum_{r=1}^{n} f(\xi_r)\delta z_r$ the limit of the sum S_n as n tents to infinity and δz_r tends to zero if exists is denoted by

$\int_a^b f(z)dz \quad or \quad \int_C f(z)dz$. This is called the line integral of f(z) along the curve C.

Closed Curve: If the points z_0 and z_n coincide then curve C is closed curve.

- The integral of closed curve is called the **contour integral** and is denoted by $\oint_C f(z)dz$

Relation between real and complex line integrals: If Z= x+iy so that dz=dx+idy and f(z)= u(x,y) +iv(x,y) then the complex line integral $\int_C f(z)dz$ can be expressed as sum or difference of two line integrals of real functions as under

$$\int_C f(z)dz = \int_C (udx - vdy) + i \int_C vdx + udy = \int_C (u+iv)(dx+idy)$$

- If f(z) =1 then we have $\int_C |dz| = \int_C ds = \ell$ where ℓ is the length of the path of integration.
- If C is a closed curve then $\int_C dz$ =0
- If C is a circle of radius r and center z_0 and if n is an integer then $\int_C \frac{dz}{(z-z_0)^{n+1}} = 0, \quad n \neq 0$

  = 2Π i, n=0

Essay Questions:

1. Prove that $\int\limits_C \frac{dz}{(z-a)} = 2\pi i,$

2. Prove that $\int\limits_C (z-a)^n dz = 0$ where n is any integer $\neq$ -1 and C is a circle |z-a|=r

3. Evaluate $\int\limits_0^{1+i} (x-y+ix^2)dz$ along the line z= 0 to z =1+I

4. Integrate f(z) =x^2+ixy from A(1,1) to B(2,8) along the straight line AB

5. Evaluate $\int (2y+x^2)dx+(3x-y)dy$ along the parabola x=2t, y=t^2+3 joining the points (0,3) and (2,4)

6. Evaluate $\int (2y+x^2)dx+(3x-y)dy$ along the parabola x=2t y=t^2+3 joining the points (0,3) and (2,4)

Simply Connected Region: A region is said to be simply connected if any simple closed curve lying in R can be shrunk to a point with out leaving R

Multiply connected region: A region that is not simply connected is called multiply connected region.

Cauchy's Integral Theorem: If f(z) is analytic function and f^1(z)is continuous at each point with in or on a closed curve C then $\int\limits_C f(z)dz$ =0

Extension of Cauchy's Integral theorem: If C_1 and C_2 are two simple closed curves and if C_2 lies entirely within the closed region between C_1 and C_2 then $\int\limits_{C_1} f(z)dz$ = $\int\limits_{C_2} f(z)dz$ both the integrals are taken in the same direction

If there are finite number of contours C_1,C_2……C_n with in C and f(z) is analytic in the region with in the region between C_1,C_2……C_n then we have

$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots\ldots + \int_{C_n} f(z)dz$ provided all the integrals are taken in same direction.

Cauchy's Integral formula: If f(z) is an analytic function inside and on a simple closed curve C and z_0 is any point within C then $f(z_0) = \frac{1}{2i\pi}\int \frac{f(z)}{(z-z_0)}dz$

Derivative of an Analytic function: $f(z_0) = \frac{n!}{2i\pi}\int \frac{f(z)}{(z-z_0)^{n+1}}dz$

Cauchy's Inequality: If $|f(z)| \leq M$ along C the circle $|z-z_0| = 2$ then $|f^n(z_0)| \leq n!\, M/r^n$ where n= 0,1,2,….

Liouville's theorem: If f(z) is analytic in the whole Z- plane and if |f(z)| is bounded for allz then f(z) must be constant

Essay Questions

1. Evaluate $\int_{C} \frac{2z^2+z}{z^2-1}dz$ where a) C is a circle $|z-1| = 1$ b) C is a circle $|z|=2$

2. If f(a) = $\int_{C} \frac{3z^2+7z+1}{z-a}dz$ where c is $|z| = 2$. Find f(3), f(1) f(1-I) $f^{11}(1-I)$

3. Evaluate $\oint_{c} \frac{e^{2z}}{(z-1)(z-2)}dz$ where c is the circle $|z|=3$ using cauchy integral formula

4. State and prove Caucshy's integral theorem.

5. Establish Cauchy's integral formula.

6. Evaluate $\int_{1-i}^{2+i}(2x+iy+1)dz$ along two paths x=t+1, y=$2t^2$-1.

7. Evaluate $\int_{c} \bar{z}\, dz$ where C is

(i) the line segment joining the points(1,1) and (2,4)

(ii) the curve x=t, $y=t^2$ joining the points (1,1) and (2,4).

8. Evaluate $\int_c \frac{z+4}{z^2+2z+5}dz$ is C is (i) the circle |z+1-i|=2 (ii) the circle |z|=1

(iii)the circle |z+1+i|=2.

9. Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ where C is |z-i|=2.

10. Evaluate $\int_0^{1+i}(x-y+ix^2)dz$ along the line z= 0 to z =1+I

11. Evaluate $\int_0^{2+i} z^{-2}dz$ along the line y= $\frac{x}{2}$.

(COMPLEX POWER SERIES)

Infinite series- Taylor's and Laurent's series.

Taylor's series : If a function f(z) is Analytic inside a circle 'c' whose center is 'a' then for all z inside c f(z)=

$$f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+----+\frac{(z-a)^n}{n!}f^n(a)+---$$

1. Put z=a+h (or) h=z-a

   $\therefore$ f(a+h)= $f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+----+\frac{h^n}{n!}f^n(a)+---$

2. Put a=0 f(z)= $f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+----+\frac{z^n}{n!}f^n(0)+---$ is called Maclaurin's series.

Laurent's series:

If f(z) is analytic inside and on the boundary of the ring stated region R bounded by two concentric circles c_1 and c_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having center at 'a' then for all z in R

F(z)= $a_0 + a_1(z-a) + a_2(z-a)^2 + ----+a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ---$

Where $a_n = \frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{(w-a)^{n+1}} dw$, n=0,1,2,-------

and $a_{-1} = \frac{1}{2\pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{-n+1}} dw$,n=1,2,3,-----

Essay Questions:

1. Find the Laurents series expansion of $\frac{1}{(z+1)(z+3)}$ in powers of (z+1) for the range 0<|z+1|<2
2. Obtain the Taylor and Laurent's series which represents the function $\frac{z^n - 1}{(z+2)(z+3)}$ in the region I)

   |z|<2 (ii) 2<|Z|<3 (iii)|z|>3
3. Expand cosz in Taylor's series about z= $\frac{\pi}{2}$
4. Expand the following function in Laurents series.

   (i) $\frac{z}{(z+1)(z+2)}$ about z=-2 (ii) $\frac{e^z}{(z-1)^2}$ about z=1.

5. Represent a function f(z)= $\frac{z}{(z-1)(z-3)}$ by a series of positive and negative powers of

   (z-1) which converges to f(z) when 0<|z-1|<2.

6. Expand f(z)= $\frac{1}{(z-1)(z-2)}$ in the region 1<|z|<2.
7. Expand f(z)= $\frac{z+3}{z(z^2 - z - 2)}$ in the region(i) |z|=1 (ii) 1<|z|<2.

Zeros and Singularities:

Zeros of an Analytic function: A zero of an Analytic function f(z) is that value of z for which f(z)=0

Singularities of an Analytic function : A singularity of a function is that point at which the function f(z) ceases to be analytic.

Isolated Singularity : If z=a is a singularity of f(z) and if f(z) is analytic at each point in its neighbourhood then z=a is called an isolated singularity.

Removable singularity f(z)= $\sum_{n=0}^{\infty} a_n (z-a)^n$ The singularity can be removed by defining the function f(z) at z=a in such a way that it becomes analytic at z=a.

Poles: If all the negative powers of (z-a) in

f(z)= $a_0 + a_1(z-a) + a_2(z-a)^2 + ----+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ---$

after the n[th] we missing then the singularity at z=a is called a pole of order n.

A pole of first order is called a Simple pole.

<u>Essential Singularity</u>: If the number of negative powers of (z-a) in

f(z)= $a_0 + a_1(z-a) + a_2(z-a)^2 + ----+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ---$

is in finite then z=a is called an essential singularity in this case $\lim_{z \to a} f(z)$ does not exist .

<u>Residues</u>: The coefficient of (z-a) in the expansion of f(z) around an isolated singularity is called the residue of f(z) at that point and is written as $\operatorname{Res}_{z=a} f(z)$

<u>Evaluation of Residues</u>:

1. If f(z) has a simple pole at z=a then $\operatorname{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z)$

2. Suppose f(z) = $\frac{\phi(z)}{\psi(z)}$ where $\psi(z) = (z-a)F(z)$ where F(a)$\neq$0 then $\operatorname*{Res}_{z=a} f(z)$ = $\frac{\phi(a)}{\psi'(a)}$

3. Let z=a be a pole of f(z) of order m then

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{\angle(m-1)} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

Cauchy's Residue theorem:

If f(z) is analytic in a closed curve C except at a finite number of singular points within C then $\int_C f(z)dz$ =

2iΠ(Sum of the residues at the singular point within C)

UNIT-III

Evaluation of some of the definite integrals:

Many of the definite integrals can be evaluated by using cauchy's

residue theorem. It may be observed that a definite integral that can be evaluated by using Cauchy's residue theorem may also be evaluated by other methods namely:

I) Integration around the unit circle

II) Integration of the type $\int_{-\infty}^{\infty} f(x)dx$

III) Indenting contours having poles on real axis

IV) Using Jordan's Lemma

Essay Questions:

7. Find the kind of singularities of $\frac{\cos \pi z}{(z-a)^3}$ at z=0 and z= ∞

8. Find the residue of the following functions at each of the poles:

i) $\dfrac{4z-3}{z(z-1)(z-2)}$ ii) $\dfrac{1-e^{2z}}{z^4}$

9. The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as |z| tends to infinity and if f(2)=5 and f(-1)=2 then find f(z)

10. Find the residue of $\dfrac{z^2}{z^4-1}$ at those singular points which lie in side the circle |z|=2

11. Evaluate $\int_C \dfrac{3z-4}{z(z-1)(z-2)}dz$ where C is a circle |z| =3/2

12. Find the residue of $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ at the respective poles

13. Show that $\int_0^{2\pi} \dfrac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \dfrac{d\theta}{a+b\sin\theta} = \dfrac{2\pi}{\sqrt{a^2-b^2}}$ (a>b>0)

14. Evaluate $\int_0^{2\pi} \dfrac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2}$ (0 < a <1) by using contour integration

15. Evaluate $\int_{-\infty}^{\infty} \dfrac{dx}{(x^2+a^2)(x^2+b^2)}$ (a >0 , b>0)

16. Using the complex variable technique evaluate $\int_0^{\infty} \dfrac{dx}{x^6+1}$

17. Evaluate by contour integrals on $\int_0^{\infty} \dfrac{\cos mxdx}{x^2+a^2}$ a >0

18. Evaluate $\int_{-\infty}^{\infty} \dfrac{\sin xdx}{(x^2+4x+5)}$ by contour integration

19. State and prove Cauchy's Residue theorem

20. Use residue theorem evaluate $\int_c \dfrac{1}{z^2(z+2)}dz$ where C is the circle |z|=1

15. Determine the poles of the function f(z)= $\dfrac{z^2}{(z-1)^2(z+2)}$ and residue at each pole.

16. Find the residues of $\dfrac{ze^z}{(z^2+a^2)}$ at its poles.

17. Evaluate $\int_c \frac{3z-4}{z(z-1)(z-2)}dz$ where C is the circle $|z|=\frac{3}{2}$

18. a) Evaluate $\int_c z^2 e^{1/z} dz$ C: $|z|=1$ b) Evaluate $\int_c \frac{e^z}{z^2+1} dz$ over the circular path $|z|=2$.

19. Find the residue of the following at the respective poles.

(i) $\frac{z}{(z^2+1)}$ (ii) $\frac{z^2-2z}{(z+1)^2(z^2+4)}$

20. Show that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$ by using residues.

21. Evaluate the following integrals by contour integration.

(i) $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta + a^2}$ 0<a<1 (ii) $\int_0^{2\pi} \frac{a\sin^2\theta}{a+b\cos\theta} d\theta$

(iii) $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$ (iv) $\int_0^{\infty} \frac{dx}{x^4+16}$

(v) $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^3}$ (vi) $\int_0^{\infty} \frac{\sin mx dx}{x(x^2+a^2)}$ (a>0)

UNIT - IV

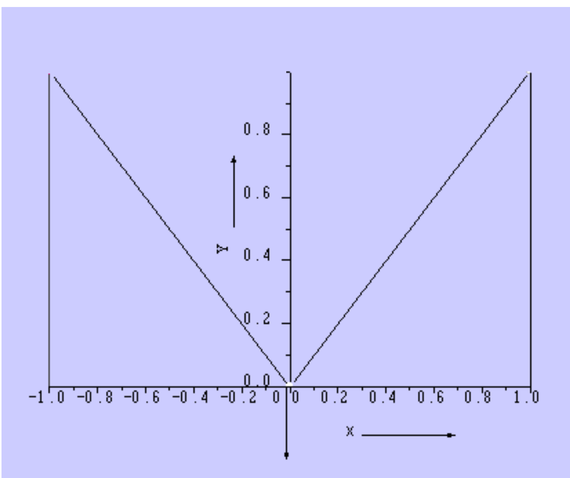
FOURIER SERIES AND TRANSFORMS

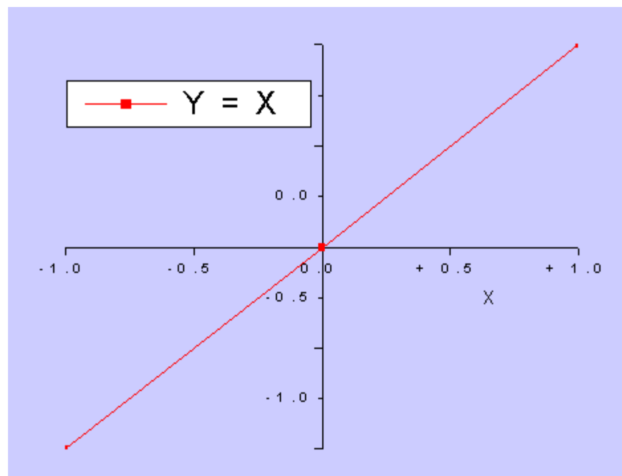
DEFINITIONS :

A function y = f(x) is said to be <u>even</u>, if f(-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y=f(x) is said to be <u>odd</u>, if f(-x) = - f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = $|x|$ in [-1,1] is even as f(-x) = $|-x| = |x|$ = f(x) and the function f(x) = x in [-1,1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below :





Graph of f(x) = $|x|$

Graph of f(x) = x

Note that the graph of f(x) = $|x|$ is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then

   - h(x) = f(x) x g(x) is odd
   - h(x) = f(x) x f(x) is even
   - h(x) = g(x) x g(x) is even

For example,

1. $h(x) = x^2 \cos x$ is even, since both x^2 and cosx are even functions

2. $h(x) = x\sin x$ is even, since x and sinx are odd functions

3. $h(x) = x^2 \sin x$ is odd, since x^2 is even and sinx is odd.

2. If f(x) is even, then $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

3. If f(x) is odd, then $$\int_{-a}^{a} f(x)dx = 0$$

For example,

$$\int_{-a}^{a} \cos x dx = 2\int_{0}^{a} \cos x dx,$$ as cosx is even

and $$\int_{-a}^{a} \sin x dx = 0,$$ as sinx is odd

<u>PERIODIC FUNCTIONS</u> :-

A periodic function has a basic shape which is repeated over and over again. The fundamental range is the time (or sometimes distance) over which the basic shape is defined. The length of the fundamental range is called the period.

A general periodic function f(x) of period T satisfies the condition

$$f(x+T) = f(x)$$

Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots.. = f(x+nT)$$

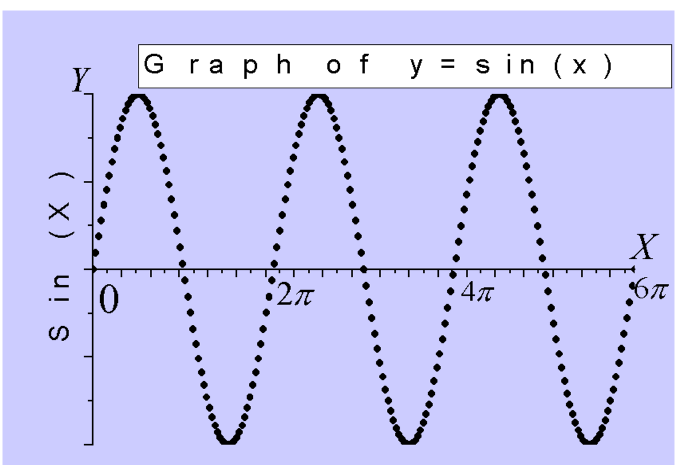
Thus,

$$f(x) = f(x+nT),\ n=1,2,3,\ldots..$$

The function f(x) = sinx is periodic of period 2π since

$$\text{Sin}(x+2n\pi) = \sin x, \quad n=1,2,3,\ldots\ldots..$$

The graph of the function is shown below :

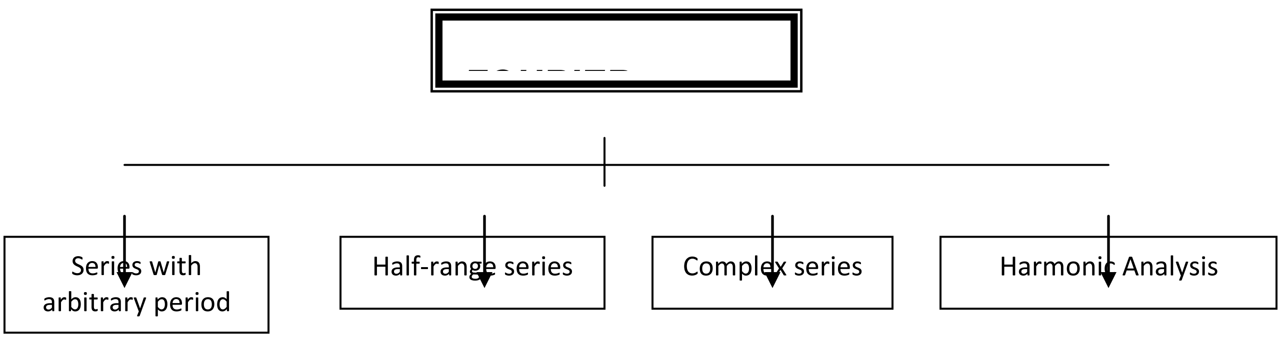


Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.
It may be verified that a linear combination of periodic functions is also periodic.

FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet's conditions :

1. f(x) is defined in an interval (a,a+2*l*), and f(x+2*l*) = f(x) so that f(x) is a periodic function of period 2*l*.
2. f(x) is continuous or has only a finite number of discontinuities in the interval (a,a+2*l*).
3. f(x) has no or only a finite number of maxima or minima in the interval (a,a+2*l*).

Also, let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,..... \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,...... \quad (3)$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval (a,a+2*l*). Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \; .... \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

Particular Cases

Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2*l*). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,......\infty \tag{6}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2*l*).

If we set *l*=π, then f(x) is defined over the interval (0,2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx, \quad n=1,2,.....\infty \tag{7}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \quad n=1,2,.....\infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{8}$$

Case (ii)

Suppose a=-*l*. Then f(x) is defined over the interval (-*l* , *l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \qquad n = 1,2,......\ \infty \tag{9}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x dx, \qquad n=1,2,\ldots\ldots\infty$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l* , *l*).

If we set *l* = π, then f(x) is defined over the interval (-π, π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad n=1,2,\ldots\ldots\infty \qquad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n=1,2,\ldots\ldots\infty$$

Putting *l* = π in (5), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

PARTIAL SUMS

The Fourier series gives the exact value of the function. It uses an infinite number of terms which is impossible to calculate. However, we can find the sum through the partial sum S_N defined as follows :

$$S_N(x) = a_0 + \sum_{n=1}^{n=N}\left[a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x\right]$$ where N takes positive integral values.

In particular, the partial sums for N=1,2 are

$$S_1(x) = a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)$$

$$S_2(x) = a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)$$

If we draw the graphs of partial sums and compare these with the graph of the original function f(x), it may be verified that $S_N(x)$ approximates f(x) for some large N.

Some useful results :

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + \ldots\ldots.$$

Here u', u'' ,….. are the successive derivatives of u and

$$v_1 = \int v\,dx, v_2 = \int v_1 dx, \ldots\ldots$$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nx\,dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}\left[a \cos bx + b \sin bx\right]$$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right]$$

3. If 'n' is integer, then

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\sin 2n\pi = 0$, $\cos 2n\pi=1$

Examples

1. Obtain the Fourier expansion of

$$f(x) = \frac{1}{2}(\pi - x) \text{ in } -\pi < x < \pi$$

We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

2. Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that

$$\cos ech\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus,

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

$$f(0)=1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

or

$$1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}\right]$$

or $$1 = \frac{2\sinh \pi}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi \operatorname{cosech} \pi = 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

This is the desired deduction.

3. Obtain the Fourier expansion of f(x) = x^2 over the interval (-π, π). Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \infty$$

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or $$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx,$$ since f(x)cosnx is even

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$
$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x)sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$
$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$

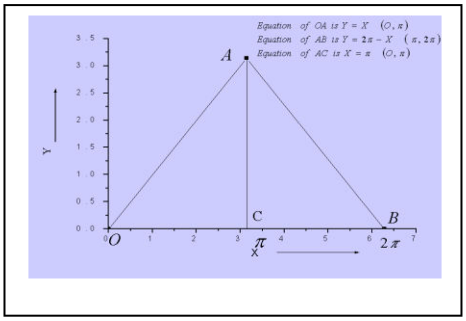
4. Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \le x \le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

The graph of f(x) is shown below.



Here OA represents the line f(x)=x, AB represents the line f(x)=(2π-x) and AC represents the line x=π. Note that the graph is symmetrical about the line AC, which in turn is parallel to y-axis. Hence the function f(x) is an even function.

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x dx = \pi$$

since f(x)cosnx is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$, since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$$

or $$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

This is the series as required.

5. Obtain the Fourier expansion of

f(x) = $\begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is

$$f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1-2(-1)^n\right]}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+)=0$, $f(x^-)=-\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0-\pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

6. Obtain the Fourier series of $f(x) = 1-x^2$ over the interval (-1,1).

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_{0}^{1}(1-x^2)dx = 2\left[x-\frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$ as $f(x)\cos(n\pi x)$ is even

$$= 2\int_{0}^{1}(1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)+(-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$ =0, since $f(x)\sin(n\pi x)$ is odd.

The Fourier series of f(x) is

$$f(x) = \frac{2}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

7. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1+\frac{4x}{3} in -\frac{3}{2} < x \le 0 \\ 1-\frac{4x}{3} in 0 \le x < \frac{3}{2} \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

The period of f(x) is $\frac{3}{2}-\left(\frac{-3}{2}\right)=3$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)-\left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\Bigg|_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also,

$$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

putting x=0, we get

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or

$$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

Thus,

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

NOTE

Here verify the validity of Fourier expansion through partial sums by considering an example. We recall that the Fourier expansion of $f(x) = x^2$ over $(-\pi, \pi)$ is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$

Let us define

$$S_N(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{n=N}\frac{(-1)^n \cos nx}{n^2}$$

The partial sums corresponding to N = 1,2,…..6 are

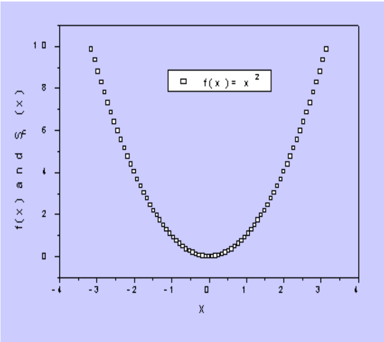
$$S_1(x) = \frac{\pi^2}{3} - 4\cos x$$

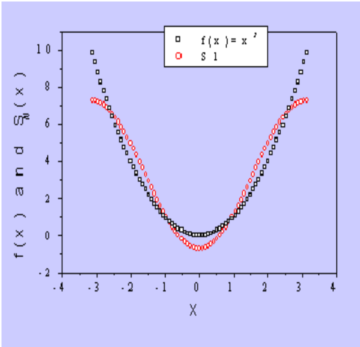
$$S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$$

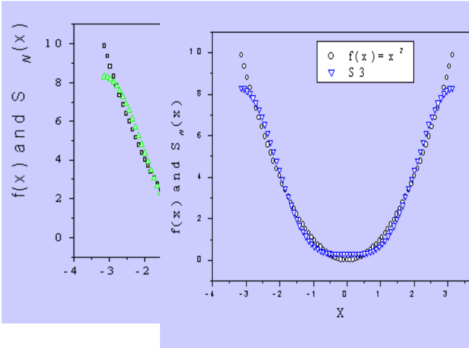
$$\dots$$

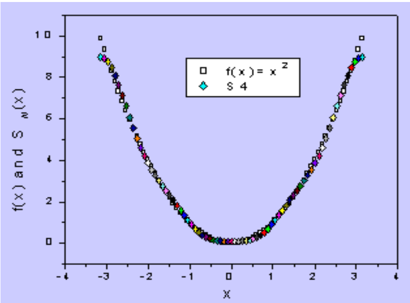
$$S_6(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x + \frac{1}{9}\cos 5x$$

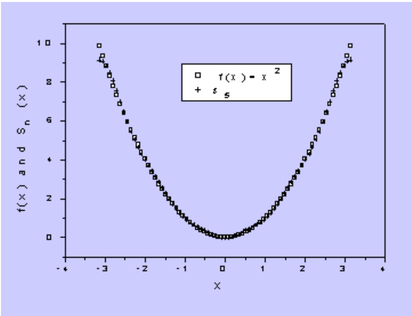
The graphs of S_1 , S_2 , …S_6 against the graph of $f(x) = x^2$ are plotted individually and shown below :











On comparison, we find that the graph of $f(x) = x^2$ coincides with that of S_6 (x). This verifies the validity of Fourier expansion for the function considered.

Exercise

Check for the validity of Fourier expansion through partial sums along with relevant graphs for other examples also.

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period 2*l* may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0,*l*) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose f(x) = φ(x) is given in the interval (0,*l*). Then we define f(x) = -φ(-x) in (-*l*,0). Hence f(x) becomes an odd function in (-*l* , *l*). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over (0,*l*).

Putting l=π in (11), we obtain the half-range sine series of f(x) over (0,π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Cosine series :

Let us define

$$f(x)=\begin{cases}\phi(x) & \text{in } (0,l) \quad .....given \\ \phi(-x) & \end{cases}$$

in (-*l*,0) …..in order to make the function even.

Then the Fourier series of f(x) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,

$$a_0=\frac{2}{l}\int_0^l f(x)dx$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over (0,*l*)

Putting l = π in (12), we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

where

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{n}=1,2,3,\ldots..$$

Examples :

1. Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

We have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1-(-1)^n\right]\sin nx$$

2. Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \qquad over(0,\pi)$$

Here

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

f(x) = $\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$

3. Obtain the half-range cosine series of f(x) = c-x in 0<x<c

Here

$$a_0 = \frac{2}{c}\int_0^c (c - x)\,dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c - x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n = \frac{2c}{n^2\pi^2}\left[1 - (-1)^n\right]$$

The cosine series is given by

f(x) = $\frac{c}{2}+\frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$

Exercices:

Obtain the Fourier series of the following functions over the specified intervals :

1. f(x) = $x+\frac{x^2}{4}$ over (-π, π)

2. f(x) = 2x + $3x^2$ over (-π, π)

3. f(x) = $\left(\frac{\pi - x}{2}\right)^2$ over (0, 2π)

4. f(x) = x over (-π, π) ; Deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-.......\infty$

5. f(x) = $|x|$ over (-π, π) ; Deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+.......\infty$

6. f(x) = $\begin{cases}\pi + x, -\pi \le x < 0\\ \pi - x, 0 \le x < \pi\end{cases}$ over (-π, π)

   Deduce that

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+.......\infty$$

7. f(x) = $\begin{cases}-1, -\pi < x < 0\\ 0, x = 0\\ 1, 0 < x < \pi\end{cases}$ over (-π, π)

   Deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-.......\infty$

8. f(x) = x sinx over $0 \le x \le 2\pi$; Deduce that

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}=\frac{3}{4}$$

9. f(x) = $\begin{cases} 0, -2 \le x \le 0 \\ a, 0 < x \le 2 \end{cases}$ over (-2 , 2)

10. f(x) = x(2-x) over (0,3)

11. f(x) = x^2 over (-1,1)

12. f(x) = $\begin{cases} \pi x, 0 \le x \le 1 \\ \pi(2-x), 1 \le x \le 2 \end{cases}$

Obtain the half-range sine series of the following functions over the specified intervals :

13. f(x) = cosx over $(0,\pi)$

14. f(x) = $\sin^3 x$ over $(0,\pi)$

15. f(x) = $lx-x^2$ over $(0 , l)$

Obtain the half-range cosine series of the following functions over the specified intervals :

16. f(x) = x^2 over $(0,\pi)$

17. f(x) = xsinx over $(0,\pi)$

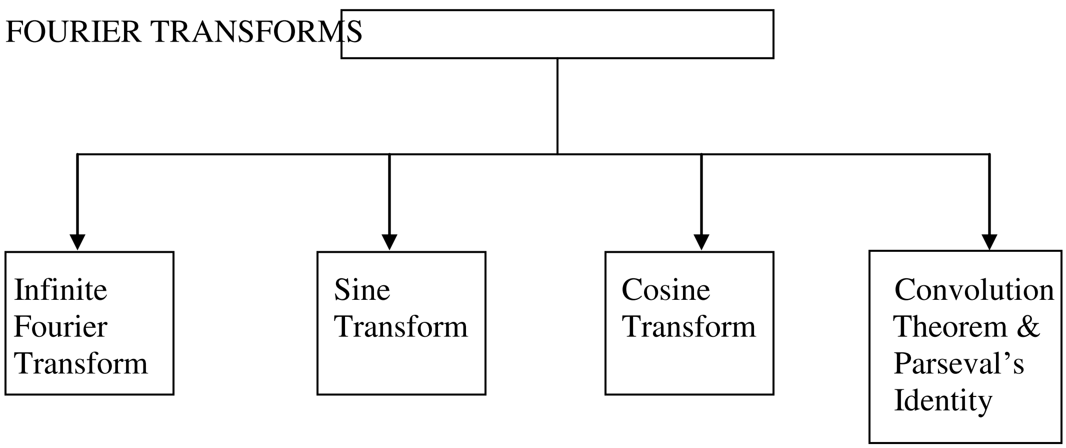
18. f(x) = $(x-1)^2$ over (0,1)

19. f(x) = $\begin{cases} kx, 0 \le x \le \frac{l}{2} \\ k(l-x), \frac{l}{2} \le x \le l \end{cases}$

FOURIER TRANSFORMS

Introduction

Fourier Transform is a technique employed to solve ODE's, PDE's,IVP's, BVP's and Integral equations.
The subject matter is divided into the following sub topics :



Infinite Fourier Transform

Let f(x) be a real valued, differentiable function that satisfies the following conditions:
1) f(x) and its derivative $f'(x)$ are continuous , or have only a finite number of simple discontinuities in every finite interval, and

2) the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

Also, let α be non - zero real parameter. The infinite Fourier Transform of f(x) is defined by

$$\hat{f}(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

provided the integral exists.

The infinite Fourier Transform is also called complex Fourier Transform or just the Fourier Transform. The inverse Fourier Transform of $\hat{f}(\alpha)$ denoted by $F^{-1}[\hat{f}(\alpha)]$ is defined by

$$F^{-1}[\hat{f}(\alpha)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$$

Note : The function f(x) is said to be self reciprocal with respect to Fourier transform

if $\hat{f}(\alpha) = f(\alpha)$.

Basic Properties

Below we prove some basic properties of Fourier Transforms:

1. **Linearity Property**

For any two functions f(x) and ϕ(x) (whose Fourier Transforms exist) and any two constants a and b,

$$F[af(x)+b\phi(x)] = aF[f(x)]+bF[\phi(x)]$$

Proof

By definition, we have

$$F[af(x)+b\phi(x)] = \int_{-\infty}^{\infty}[af(x)+b\phi(x)]\,e^{i\alpha x}dx$$

$$= a\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx + b\int_{-\infty}^{\infty}\phi(x)\,e^{i\alpha x}dx$$
$$= aF[f(x)]+bF[\phi(x)]$$

This is the desired property.

In particular, if a = b = 1, we get

$$F[f(x)+\phi(x)] = F[f(x)]+F[\phi(x)]$$

Again if a= -b = 1, we get

$$F[f(x)-\phi(x)]=F[f(x)]-F[\phi(x)]$$

2. **Change of Scale Property**

If $\hat{f}(\alpha)=F[f(x)]$, then for any non-zero constant a, we have

$$F[f(x)]=\frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$$

Proof : By definition, we have

$$F[f(ax)]=\int_{-\infty}^{\infty}[f(ax)]e^{i\alpha x}dx \qquad (1)$$

Suppose a > 0. let us set ax = u. Then expression (1) becomes

$$F[f(ax)]=\int_{-\infty}^{\infty}[f(u)]e^{i\left(\frac{\alpha}{a}\right)u}\frac{du}{a}$$

$$=\frac{1}{a}\hat{f}\left(\frac{\alpha}{a}\right) \qquad (2)$$

Suppose a < 0. If we set again ax = u, then (1) becomes

$$F[f(ax)]=\int_{\infty}^{-\infty}[f(u)]e^{i\alpha\left(\frac{u}{a}\right)}\frac{du}{a}$$

$$=-\frac{1}{a}\int_{-\infty}^{\infty}[f(u)]e^{i\left(\frac{\alpha}{a}\right)u}du$$

$$=-\frac{1}{a}\hat{f}\left(\frac{\alpha}{a}\right) \qquad (3)$$

Expressions (2) and (3) may be combined as

$$F[f(ax)]=\frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$$

This is the desired property

3. **Shifting Properties**

For any real constant 'a',

$$(i)\quad F[f(x-a)]=e^{i\alpha a}\hat{f}(\alpha)$$

$(ii)\ F\left[e^{iax} f(x)\right] = \hat{f}(\alpha + a)$

<u>Proof</u> : (i) We have

$$F\left[f(x)\right] = \hat{f}(\alpha) = \int_{-\infty}^{\infty} \left[f(x)\right] e^{i\alpha x} dx$$

Hence, $F\left[f(x-a)\right] = \int_{-\infty}^{\infty} \left[f(x-a)\right] e^{i\alpha x} dx$

Set x-a = t. Then dx = dt.Then,

$$F\left[f(x-a)\right] = \int_{-\infty}^{\infty} \left[f(t)\right] e^{i\alpha(t+a)} dt$$

$$= e^{i\alpha a} \int_{-\infty}^{\infty} \left[f(t)\right] e^{i\alpha t} dt$$

$$= e^{i\alpha a}\ \hat{f}(\alpha)$$

ii) We have

$$\hat{f}(\alpha + a) = \int_{-\infty}^{\infty} f(x)\, e^{i(\alpha+a)x} dx$$

$$= \int_{-\infty}^{\infty} \left[f(x) e^{iax}\right] e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} g(x) e^{iax}\ dx,\ \ where\ g(x) = f(x) e^{iax}$$

$$= F\left[g(x)\right]$$

$$= F\left[e^{iax} f(x)\right]$$

This is the desired result.

4. **Modulation Property**

$$If \quad F[f(x)] = \hat{f}(\alpha),$$

$$then,\ F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)\right]$$

where 'a' is a real constant.

Proof : We have

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

Hence

$$F[f(x)\cos ax] = F\left[f(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)\right], \text{by using linearity and shift properties.}$$

This is the desired property.

Note : Similarly

$$F[f(x)\sin ax] = \frac{1}{2}\left[\hat{f}(\alpha + a) - \hat{f}(\alpha - a)\right]$$

Examples

1. Find the Fourier Transform of the function $f(x) = e^{-a|x|}$ where $a > 0$

For the given function, we have

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a|x|} e^{i\alpha x}\, dx$$

$$= \left[\int_{-\infty}^{0} e^{-a|x|} e^{i\alpha x}\, dx + \int_{0}^{\infty} e^{-a|x|} e^{i\alpha x}\, dx\right]$$

Using the fact that $|x| = x,\ 0 \le x < \infty$ and $|x| = -x,\ -\infty < x \le 0$, we get

$$F[f(x)] = \left[\int_{-\infty}^{0} e^{ax} e^{i\alpha x}\ dx + \int_{0}^{\infty} e^{-ax} e^{i\alpha x}\ dx\right]$$

$$= \left[\int_{-\infty}^{0} e^{(a+i\alpha)x}\ dx + \int_{0}^{\infty} e^{-(a-i\alpha)x}\ dx\right]$$

$$= \left[\left\{\frac{e^{(a+i\alpha)x}}{(a+i\alpha)}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(a-i\alpha)x}}{-(a-i\alpha)}\right\}_{0}^{\infty}\right]$$

$$= \left[\frac{1}{(a+i\alpha)} + \frac{1}{(a-i\alpha)}\right]$$

$$= \left[\frac{2a}{(a^2+\alpha^2)}\right]$$

2. Find the Fourier Transform of the function

$$f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

where ‘a’ is a positive constant. Hence evaluate

$$(i)\quad \int_{-\infty}^{\infty} \frac{\sin \alpha\, a\ \cos \alpha\, x}{\alpha}\, d\alpha$$

$$(ii)\quad \int_{0}^{\infty} \frac{\sin \alpha}{\alpha} d\alpha$$

For the given function, we have

$$F[f(x)] = \left[\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx\right]$$

$$= \left[\int_{-\infty}^{-a} f(x) e^{i\alpha x} dx + \int_{-a}^{a} f(x) e^{i\alpha x} dx + \int_{a}^{\infty} f(x) e^{i\alpha x} dx\right]$$

$$= \left[\int_{-a}^{a} e^{i\alpha x} dx\right]$$

$$= 2\left[\frac{\sin \alpha a}{\alpha}\right]$$

$$Thus \quad F[f(x)] = \hat{f}(\alpha) = 2\left(\frac{\sin \alpha\ a}{\alpha}\right) \qquad (1)$$

Inverting $\hat{f}(\alpha)$ by employing inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2[\ \frac{\sin \alpha a}{\alpha}\ ]e^{-i\alpha x}\,d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a(\cos \alpha x - i\sin \alpha x)}{\alpha}\,d\alpha$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin \alpha a(\cos \alpha\ x)}{\alpha}\,d\alpha - i\int_{-\infty}^{\infty} \frac{\sin \alpha\ a \sin \alpha x}{\alpha}\,d\alpha\right]$$

Here, the integrand in the first integral is even and the integrand in the second integral is odd. Hence using the relevant properties of integral here, we get

$$f(x) = \tfrac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\,d\alpha$$

or

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\,d\alpha = \pi f(x)$$

$$= \begin{cases} \pi, & |x| \le a \\ 0, & |x| > a \end{cases}$$

For $x = 0, a = 1$, this yields

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha}\,d\alpha = \pi$$

Since the integrand is even, we have

$$2\int_{0}^{\infty} \frac{\sin \alpha}{\alpha}\,d\alpha = \pi$$

or

$$\int_{0}^{\infty} \frac{\sin \alpha}{\alpha}\,d\alpha = \tfrac{\pi}{2}$$

3. Find the Fourier Transform of $f(x) = e^{-a^2x^2}$ where 'a' is a positive constant.

Deduce that $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier Transform.

Here

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2 - i\alpha x)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{i\alpha}{2a}\right)^2 + \frac{\alpha^2}{4a^2}\right]} dx$$

$$= e^{-\left(\alpha^2/4a^2\right)} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2} dx$$

Setting $t = ax - \frac{i\alpha}{2a}$, we get

$$F[f(x)] = e^{-\left(\alpha^2/4a^2\right)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{1}{a} e^{-\left(\alpha^2/4a^2\right)} 2\int_{0}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a} e^{-\left(\alpha^2/4a^2\right)} \sqrt{\pi}, \text{using gamma function.}$$

$$\hat{f}(\alpha) = \frac{\sqrt{\pi}}{a} e^{-\left(\alpha^2/4a^2\right)}$$

This is the desired Fourier Transform of f(x).

For $a^2 = \frac{1}{2}$ in $f(x) = e^{-a^2x^2}$

we get $f(x) = e^{-x^2/2}$ and hence,

$\hat{f}(\alpha) = \sqrt{2\pi}\, e^{-\alpha^2/2}$

Also putting $x = \alpha$ in $f(x) = e^{-x^2/2}$, we get $f(\alpha) = e^{-\alpha^2/2}$.

Hence, $f(\alpha)$ and $\hat{f}(\alpha)$ are same but for constant multiplication by $\sqrt{2\pi}$.

Thus $f(\alpha) = \hat{f}(\alpha)$

It follows that $f(x) = e^{-x^2/2}$ is self reciprocal

ASSIGNMENT

Find the Complex Fourier Transforms of the following functions :

(1) $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ where 'a' is a positive constant

(2) $f(x) = \begin{cases} 0, & x < a \\ 1, & a \le x \le b \\ 0, & x > b \end{cases}$ where 'a' and 'b' are positive constants

(3) $f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

(4) $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

(5) $f(x) = xe^{-a|x|}$ where 'a' is a positive constant

(6) $f(x) = e^{-|x|}$

(7) $f(x) = \cos 2x^2$

(8) $f(x) = \sin 3x^2$

(9) Find the inverse Fourier Transform of $\hat{f}(\alpha) = e^{-\alpha^2}$

FOURIER SINE TRANSFORMS

Let f(x) be defined for all positive values of x.

The integral $\int_0^\infty f(x) \sin \alpha x dx$ is called the Fourier Sine Transform of f(x). This is denoted by $\hat{f}_s(\alpha)$ *or* $F_s[f(x)]$. *Thus*

$$\hat{f}_s(\alpha) = F_s[f(x)] = \int_0^\infty f(x) \sin \alpha x \, dx$$

The inverse Fourier sine Transform of $\hat{f}_s(\alpha)$ is defined through th e integral $\frac{2}{\pi} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, d\alpha$

This is denoted by f(x) or $F_s^{-1}[f_s(\alpha)]$. *Thus*

$$f(x) = F_s^{-1}[f_s(\alpha)] = \frac{2}{\pi} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, d\alpha$$

Properties

The following are the basic properties of Sine Transforms.

(1) **LINEARITY PROPERTY**

If ‘a’ and ‘b’ are two constants, then for two functions f(x) and ϕ(x), we have

$$F_s[af(x) + b\phi(x)] = aF_s[f(x)] + bF_s[g(x)]$$

Proof : By definition, we have

$$F_s[af(x)+b\phi(x)] = \int_0^\infty [af(x)+b\phi(x)]\sin \alpha x\, dx$$

$$= aF_s[f(x)] + bF_s[\phi(x)]$$

This is the desired result. In particular, we have

$$F_s[f(x)+\phi(x)] = F_s[f(x)] + F_s[\phi(x)]$$

and

$$F_s[f(x)-\phi(x)] = F_s[f(x)] - F_s[\phi(x)]$$

(2) **CHANGE OF SCALE PROPERTY**

$$\text{If } F_s[f(x)] = \hat{f}_s(\alpha), \text{ then for } a \neq 0, \text{ we have}$$

$$F_s[f(ax)] = \frac{1}{a}\hat{f}_s\left(\frac{\alpha}{a}\right)$$

Proof : We have

$$F_s[f(ax)] = \int_0^\infty f(ax)\sin \alpha\, x\, dx$$

Setting ax = t , we get

$$F_s[f(ax)] = \int_0^\infty f(t)\sin\left(\frac{\alpha}{a}\right)t\left(\frac{dt}{a}\right)$$

$$= \frac{1}{a}\hat{f}_s\left(\frac{\alpha}{a}\right)$$

(3) **MODULATION PROPERTY**

$$\text{If } F_s[f(x)] = \hat{f}_s(\alpha), \text{ then for } a \neq 0, \text{ we have}$$

$$F_s[f(x)\cos ax] = \tfrac{1}{2}\left[\hat{f}_s(\alpha+a) + \hat{f}_s(\alpha-a)\right]$$

Proof : We have

$$F_s\left[f(x)\cos ax\right]=\int_0^{\infty} f(x)\cos ax\sin \alpha x\, dx$$

$$=\frac{1}{2}\left[\int_0^{\infty} f(x)\{\sin(\alpha+a)x+\sin(\alpha-a)x\}dx\right]$$

$$=\frac{1}{2}\left[\hat{f}_s(\alpha+a)+\hat{f}_s(\alpha-a)\right],\text{ by using Linearity property.}$$

EXAMPLES

1. Find the Fourier sine transform of

$$\mathrm{f}(\mathrm{x})=\begin{cases}1, & 0\le x\le a\\ 0, & x>a\end{cases}$$

For the given function, we have

$$\hat{\mathrm{f}}_{\mathrm{s}}(\alpha)=\left[\int_0^{a}\sin\alpha\, x\, dx+\int_a^{\infty} 0\sin\alpha\, x\, dx\right]$$

$$=\left[\frac{-\cos\alpha x}{\alpha}\right]_0^a$$

$$=\left[\frac{1-\cos\alpha a}{\alpha}\right]$$

2. Find the Fourier sine transform of $\mathrm{f(x)}=\dfrac{\mathrm{e}^{-\mathrm{ax}}}{\mathrm{x}}$

Here

$$\hat{\mathrm{f}}_{\mathrm{s}}(\alpha)=\left[\int_0^{\infty}\frac{e^{-ax}\sin\alpha x\, dx}{x}\right]$$

Differentiating with respect to α, we get

$$\frac{d}{d\alpha}\hat{f}_s(\alpha) = \frac{d}{d\alpha}\left[\int_0^\infty \frac{e^{-ax}\sin \alpha x\, dx}{x}\right]$$

$$= \int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial\alpha}(\sin \alpha x)\, dx$$

performing differentiation under the integral sign

$$= \int_0^\infty \frac{e^{-ax}}{x} x\cos \alpha x\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+\alpha^2}\{-a\cos \alpha x + \alpha \sin \alpha x\}\right]_0^\infty$$

$$= \frac{a}{a^2+\alpha^2}$$

Integrating with respect to α, we get

$$\hat{f}_s(\alpha) = \tan^{-1}\frac{\alpha}{a} + c$$

But $\hat{f}_s(\alpha) = 0$ *when* $\alpha = 0$

$\therefore$ c=0

$$\hat{f}_s(\alpha) = \tan^{-1}\left(\frac{\alpha}{a}\right)$$

3. Find f(x) from the integral equation

$$\int_0^\infty f(x)\sin \alpha x dx = \begin{cases} 1, & 0 \le \alpha \le 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$$

Let $\phi(\alpha)$ be defined by

$$\phi(\alpha)=\begin{cases}1, & 0\le\alpha\le1\\ 2, & 1\le\alpha<2\\ 0, & \alpha\ge2\end{cases}$$

Given

$$\phi(\alpha)=\int_0^\infty f(x)\sin\alpha x\,dx=\hat{f}_s(\alpha)$$

Using this in the inversion formula, we get

$$\mathrm{f}(\mathrm{x})=\frac{2}{\pi}\int_0^\infty \phi(\alpha)\sin\alpha x\,dx$$

$$=\frac{2}{\pi}\left[\int_0^1\phi(\alpha)\sin\alpha x\,d\alpha+\int_1^2\phi(\alpha)\sin\alpha x\,d\alpha+\int_2^\infty\phi(\alpha)\sin\alpha x\,d\alpha\right]$$

$$=\frac{2}{\pi}\left[\int_0^1\sin\alpha x\,d\alpha\;+\int_1^2 2\sin\alpha x\,d\alpha+0\right]$$

$$=\frac{2}{\pi x}\left[1+\cos x-2\cos 2x\right]$$

ASSIGNMENT

Find the sine transforms of the following functions

$$(1)\,f(x)=\begin{cases}x, & 0<x<1\\ a-x, & 1<x<a\\ 0, & x>a\end{cases}$$

$$(2)\,f(x)=xe^{-ax},a>0$$

$$(3)\,f(x)=\begin{cases}\sin x, & 0<x<a\\ 0, & x>a\end{cases}$$

(4) Solve for f(x) given

$$\int_0^\infty f(x)\sin \alpha x\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Find the inverse sine transforms of the following functions :

(5) $\hat{f}_s(\alpha) = \dfrac{e^{-a\alpha}}{\alpha}, a > 0$

(6) $\hat{f}_s(\alpha) = \frac{\pi}{2}$

FOURIER COSINE TRANSFORMS

Let f(x) be defined for positive values of x. The integral $\int_0^\infty f(x)\cos \alpha x\, dx$

is called the Fourier Cosine Transform of f(x) and is denoted by $\hat{f}_c(\alpha)$ or $F_c[f(x)]$. Thus

$$\hat{f}_c(\alpha) = F_c[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos \alpha x dx$$

The inverse Fourier Cosine Transform of $\hat{f}_c(\alpha)$ is defined through the integral $\frac{2}{\pi}\int_0^\infty \hat{f}_c(\alpha)\cos \alpha\, x\, d\, \alpha$. This is denoted by $f(x)$ or $F_c^{-1}\left[\hat{f}_c(\alpha)\right]$. Thus

$$f(x) = F_c^{-1}\left[\hat{f}(\alpha)\right] = \frac{2}{\pi}\int_0^\infty \hat{f}_c(\alpha)\cos \alpha\, x\, d\, \alpha$$

Basic Properties

The following are the basic properties of cosine transforms :

(1) **Linearity property**

If 'a' and 'b' are two constants, then for two functions f(x) and ϕ(x), we have
$F_c[af(x)+b\phi(x)] = aF_c(f(x)) + bF_c(\phi(x))$

(2) **Change of scale property**

If $F_c\{f(x)\} = \hat{f}_c(\alpha)$, then for $a \neq 0$, we have

$$F_c[f(ax)] = \frac{1}{a}\hat{f}_c\left(\frac{\alpha}{a}\right)$$

(3) **Modulation property**

If $F_c\{f(x)\} = \hat{f}_c(\alpha)$, then for $a \neq 0$, we have

$$F_c[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}_c(\alpha + a) + \hat{f}_c(\alpha - a)\right]$$

The proofs of these properties are similar to the proofs of the corresponding properties of Fourier Sine Transforms.

Examples

(1) Find the cosine transform of the function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

We have

$$\hat{f}_c(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx$$

$$= \left[\int_0^1 x\cos\alpha x\,dx + \int_1^2 (2-x)\cos\alpha x\,dx + \int_2^\infty 0\cos\alpha x\,dx\right]$$

Integrating by parts, we get

$$\hat{f}_c(\alpha) = \left[\left\{x\left(\frac{\sin\alpha x}{\alpha}\right) - \left(\frac{-\cos\alpha x}{\alpha^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin\alpha x}{\alpha}\right) - (-1)\left(\frac{-\cos\alpha x}{\alpha^2}\right)\right\}_1^2\right]$$

$$= \left[\frac{2\cos\alpha - \cos 2\alpha - 1}{\alpha^2}\right]$$

(2) Find the cosine transform of $f(x) = e^{-ax}, a > 0$. Hence evaluate $\int_0^\infty \frac{\cos kx}{x^2 + a^2}dx$

Here

$$\hat{\mathrm{f}}_{\mathrm{c}}(\alpha) = \int_0^\infty e^{-ax} \cos \alpha x dx$$

$$= \left[\frac{e^{-ax}}{\mathrm{a}^2 + \alpha^2} \{-a \cos \alpha x + \alpha \sin \alpha x\} \right]_0^\infty$$

Thus

$$\hat{\mathrm{f}}_{\mathrm{c}}(\alpha) = \left(\frac{a}{a^2 + \alpha^2} \right)$$

Using the definition of inverse cosine transform, we get

$$\mathrm{f}(x) = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{\mathrm{a}^2 + \alpha^2} \right) \cos \alpha x d\alpha$$

or

$$\frac{\pi}{2a} e^{-ax} = \int_0^\infty \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha$$

Changing x to k, and α to x, we get

$$\int_0^\infty \frac{\cos kx}{\mathrm{x}^2 + a^2} dx = \frac{\pi e^{-ax}}{2a}$$

(4) Solve the integral equation

$$\int_0^\infty f(x) \cos \alpha x \, dx = e^{-a\alpha}$$

Let $\phi(\alpha)$ be defined by

$$\phi(\alpha) = e^{-a\alpha}$$

$$\text{Given} \quad \phi(\alpha) = \int_0^\infty f(x) \cos \alpha x \, dx = \hat{\mathrm{f}}_{\mathrm{c}}(\alpha)$$

Using this in the inversion formula, we get

$$\mathrm{f}(x)=\frac{2}{\pi}\int_0^{\infty}\phi(\alpha)\cos\alpha x\,d\alpha$$
$$=\frac{2}{\pi}\int_0^{\infty}e^{-a\alpha}\cos\alpha x\,d\alpha$$
$$=\frac{2}{\pi}\int_0^{\infty}\left[\frac{e^{-\alpha\alpha}}{a^2+x^2}\{-a\cos\alpha x+\alpha\sin\alpha x\}\right]_0^{\infty}$$
$$=\frac{2a}{\pi\left(a^2+x^2\right)}$$

ASSIGNMENT

Find the Fourier Cosine Transforms of the following functions :

(1) $f(x)=\begin{cases}4x, & 0<x<1\\ 4-x, & 1<x<4\\ 0, & x>4\end{cases}$

(2) $f(x)=e^{-ax^2}, a>0$

(3) $f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x>a\end{cases}$

(4) $f(x)=xe^{-ax},\ a>0$

(5) $f(x)=\dfrac{1}{1+x^2}$

(6) $f(x)=\dfrac{\cos 2x}{1+x^2}$

(7) **Solve for f(x) given**

$$\int_0^{\infty} f(x)\cos\alpha x\,dx=\begin{cases}1-\alpha, & 0\le\alpha\le 1\\ 0, & \alpha>1\end{cases}$$

(8) Show that

(i) $F_c\left[f(x)\sin\ ax\right]=\frac{1}{2}\left[\hat{f}_s(a+\alpha)+\hat{f}_s(a-\alpha)\right]$

(ii) $F_s\left[f(x)\sin\ ax\right]=\frac{1}{2}\left[\hat{f}_c(\alpha-a)-\hat{f}_c(\alpha-a)\right]$

CONVOLUTION

Let f(x) and g(x) be two functions such that $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} g(x)dx$ exist.

Then the integral

$$\int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

is called the convolution of f(x) and g(x), and is denoted by f * g. Thus

$$f * g = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Note that f * g is a function of x

Properties

$$1.\ f * g = g * f$$
$$2.\ f * (g+h) = (f * g) + (f * h)$$

Convolution Theorem

Let $\hat{f}(\alpha)$ and $\hat{g}(\alpha)$ be the Fourier Transforms of f(x) and g(x) respectively. Then $F\left[f * g\right] = \hat{f}(\alpha)\,\hat{g}(\alpha)$

The convolution theorem may also be rewritten as

$$f * g = F^{-1}\left[\hat{f}(\alpha)\,\hat{g}(\alpha)\right]$$

Parseval's Identity

A direct consequence of convolution theorem is Parseval's identity. The Parseval's identities in respect of Fourier transforms, sine transforms and cosine transforms are as indicated below :

Fourier Transforms:

$$(\text{a}) \int_{-\infty}^{\infty} \hat{f}(\alpha)\, \overline{\hat{g}(\alpha)}\, d\alpha = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$$

$$(b) \int_{-\infty}^{\infty} \left|\hat{f}(\alpha)\right|^2\, d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Fourier Sine Transforms:

$$(\text{a}) \int_{-\infty}^{\infty} \hat{f}_s(\alpha)\, \overline{\hat{g}(\alpha)}\, d\alpha = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$$

$$(b) \int_{-\infty}^{\infty} \left|\hat{f}_s(\alpha)\right|^2\, d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Fourier CosineTransforms:

$$(\text{a}) \int_{-\infty}^{\infty} \hat{f}_c(\alpha)\, \overline{\hat{g}(\alpha)}\, d\alpha = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$$

$$(b) \int_{-\infty}^{\infty} \left|\hat{f}_c(\alpha)\right|^2\, d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Examples

(1) Employ convolution theorem to find the inverse Fourier Transform of

$$\frac{1}{(\alpha^2+4)(\alpha^2+9)}$$

$$Let \;\; \hat{f}(\alpha) = \frac{1}{(\alpha^2+4)}, \hat{g}(\alpha) = \frac{1}{(\alpha^2+9)}$$

We recall the result

$$F\left[e^{-a|x|}\right] = \frac{a}{a^2+\alpha^2}$$

$$or$$

$$F^{-1}\, \frac{1}{(\alpha^2+a^2)} = \left(\frac{e^{-a|x|}}{a}\right)$$

For a=2, 3, we get

$$F^{-1}\ \frac{1}{(\alpha^2+4)} = \hat{f}(\alpha) = \left(\frac{e^{-2|x|}}{2}\right) = f(x)$$

$$F^{-1}\ \frac{1}{(\alpha^2+9)} = \hat{g}(\alpha) = \left(\frac{e^{-3|x|}}{3}\right) = g(x)$$

Convolution theorem is

$$F^{-1}\left[\hat{f}(\alpha)\ \hat{g}(\alpha)\right] = f * g = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\, e^{-2|x-t|}\ \frac{1}{3}\, e^{-3|t|}\ dt$$

$$= \frac{1}{12}\int_{-\infty}^{\infty} e^{-2|x-t|-3|t|}\, dt$$

2. Employ Parseval' s identity t o evaluate $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$

given that $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

For the given function, we have

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1)\, e^{i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\sin\alpha}{\alpha}\right]$$

Parseval’s identity for Fourier Transforms is

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left|\hat{f}(\alpha)\right|^2 d\alpha$$

or

$$\int_{-1}^{1} (1)^2 dx = \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \left[\frac{2\sin\alpha}{\alpha} \right] \right|^2 d\alpha$$

or

$$2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2\alpha}{\alpha^2} d\alpha$$

or

$$\int_{-\infty}^{\infty} \frac{\sin^2\alpha}{\alpha^2} d\alpha = \pi$$

or

$$\int_{0}^{\infty} \frac{\sin^2\alpha}{\alpha^2} d\alpha = \frac{\pi}{2}$$, as the integrand on the L.H.S. is even.

Replacing α by x, we get

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

ASSIGNMENT

1. Given that $F\left[e^{-|x|}\right] = \frac{1}{1+\alpha^2}$, employ convolution theorem

to find $F^{-1}\left[\frac{1}{(1+\alpha^2)^2}\right]$

2. Use Parseval's identity to prove the following :

(i) $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$

(ii) $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

(iii) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4}, \quad a > 0$

(iv) If $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, Prove that $\int_0^\infty \frac{(1-\cos x)^2}{x^4} dx = \frac{\pi}{6}$

UNIT-V

APPLICATIONS OF PDE

Unit-5

Applications of partial differential equations

Classification of partial differential equations of the second order:

The general second order linear partial differential equation in two independent variable is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

which can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y) = 0$$

Where A, B, C, D, E, F are all functions of x & y

A partial Differential equation of the form (1) is said to be

(i) Elliptic if $B^2 - 4AC < 0$ at a point in the (x,y) plane (Laplace eqn)

(ii) parabolic if $B^2 - 4AC = 0$ at a point the (x,y) plane (Heat equation)

(iii) hyperbolic if $B^2 - 4AC > 0$ at the point in the (x,y) plane (wave eqn)

Examples (1) Consider $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$

Here $B^2 - 4AC = 16 - 16 = 0$ Hence it is parabolic eqn

(2) Consider $x^2 u_{xx} + (1-y^2)u_{yy} = 0$; $-\infty < x < \infty$, $-1 < y < 1$

Here $B^2 - 4AC = 0^2 - 4x^2(1-y^2) < 0$ ∵ $y^2 < 1$

Hence it is an elliptic equation

(3) $(1+x^2)u_{xx} + (5+2x^2)u_{xy} + (4+x^2)u_{yy} = 0$

$B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2) = 9 > 0$

Hence it is Hyperbolic

Method of separation of variables

Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$

(or) Solve by the method of separation of variables $u_x = 2u_t + u$ where $u(x,0) = 6e^{-3x}$

1

Soln: we have to find $u(x,t) \Rightarrow \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ — (1)

subject to the condition $u(x,0) = 6e^{-3x}$

from (1)

$u(x,t) = X(x)T(t)$ — (3)

(3) is a soln of (1) & (3) must satisfy the eqn

we have $\frac{\partial u}{\partial x} = X'(x)T(t)$, $\frac{\partial u}{\partial t} = X(x)T'(t)$

using (3) & (4) in (1) we get

$X'(x)T(t) = 2X(x)T'(t) + X(x)T(t)$

$X'(x)T(t) = X(x)[2T'(t) + T(t)]$

(or) $\frac{X'(x)}{X(x)} = \frac{2T'(t)+T(t)}{T(t)}$

$\therefore \frac{X'(x)}{X(x)} = \frac{2T'(t)+T(t)}{T(t)} = \lambda$

i.e $X'(x) - \lambda X(x) = 0 \Rightarrow X(x) = Ae^{\lambda x}$

$\Rightarrow 2T'(t) + T(t) = \lambda T(t)$

$\Rightarrow T'(t) + \frac{(1-\lambda)}{2}T(t) = 0$

$T(t) = Be^{(\lambda-1)t/2}$

$u(x,t) = Ae^{\lambda x} \cdot Be^{(\lambda-1)t/2}$

i.e $u(x,t) = Ce^{\lambda x} \cdot e^{(\lambda-1)t/2}$

$u(x,0) = 6e^{-3x}$ we get

$Ce^{\lambda x} = 6e^{-3x}$

$\lambda = -3$, $C = 6$

Hence required soln is

$u(x,t) = 6e^{-3x}e^{-2t}$

i.e $u(x,t) = 6e^{-(3x+2t)}$

(2) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)g(y)$

obtain the solution satisfying $u = 0$, $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$ for all values of y.

(or) Solve $u_{xx} = u_y + 2u$ with $u(0,y) = 0$ & $\frac{\partial u(0,y)}{\partial x} = 1 + e^{-3y}$

Soln: Let $u = X(x)\,Y(y)$ be the soln of the given eqn. then

we have $\frac{\partial^2 u}{\partial x^2} = X''(x)\,Y(y)$

$$\frac{\partial u}{\partial y} = X(x)\,Y'(y)$$

then

$$X''(x)\,Y(y) = X(x)Y'(y) + 2X(x)Y(y)$$

$$\left[X''(x) - 2X(x)\right]Y(y) = X(x)\,Y'(y)$$

$$\frac{X''(x) - 2X(x)}{X(x)} = \frac{Y'(y)}{Y(y)}$$

$$\frac{X''(x) - 2X(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = \lambda$$

$$X''(x) - 2X(x) = \lambda X(x)$$

$$X''(x) - (\lambda+2)X(x) = 0$$

$$\therefore X(x) = Ae^{\sqrt{\lambda+2}\,x} + Be^{-\sqrt{\lambda+2}\,x}$$

$$Y'(y) - \lambda Y = 0 \Rightarrow \therefore Y(y) = Ce^{\lambda y}$$

Thus $u(x,y) = \left[Ae^{\sqrt{\lambda+2}\,x} + Be^{-\sqrt{\lambda+2}\,x}\right]e^{\lambda y}$ ——(1)

W.K.T $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ for $x = 0\ \forall y$

Hence in the soln we must have e^{0y} & e^{-3y}

$\therefore \lambda$ values are chosen $\lambda = 0$ & $\lambda = -3$

$\therefore \lambda = 0$ in (1)

$u = \left(Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}\right)e^{0\cdot y}$ ——(2)

$\therefore u(0,y) = 0\ \forall y$

$\therefore A + B = 0$

P.diff with x to x' in eqn (2)

$$\frac{\partial u}{\partial x} = \sqrt{2}\left(Ae^{\sqrt{2}x} - Be^{-\sqrt{2}x}\right)$$

$$\therefore \frac{\partial u}{\partial x} = 1 \quad \forall x = 0$$

$$\therefore \sqrt{2}(A-B) = 1$$

$$A - B = 1/\sqrt{2}$$

$$A + B = 0$$

$$A = \frac{1}{2\sqrt{2}} \ \& \ B = -\frac{1}{2\sqrt{2}}$$

$$u_1 = \left(\frac{1}{2\sqrt{2}} e^{\sqrt{2}x} - \frac{1}{2\sqrt{2}} e^{-\sqrt{2}x}\right) \cdot 1 = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x$$

$\therefore$ consider (1) with $\lambda = -3$

$$u = \left(Ae^{\sqrt{-1}x} + Be^{-\sqrt{-1}x}\right)e^{-3y}$$

$$u = \left(A^* \cos x + B^* \sin x\right)e^{-3y}$$

$$u(0,y) = 0 \Rightarrow A^* = 0$$

$$u = B^* \sin x \, e^{-3y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = B^* \cos x \, e^{-3y}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{x=0} = e^{-3y} \ \forall y$$

$$\therefore B^* e^{-3y} = e^{-3y} \ \forall y$$

$$y = B^* = 1$$

$$u_2(x,y) = \sin x \cdot e^{-3y}$$

$$\therefore u(x,y) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x + e^{-3y} \sin x$$

ONE DIMENSIONAL WAVE EQUATION

~~soln~~ of ~~eqn~~

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}} \quad (1) \Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where $c^2 = T/m$

T = tension in the string at the any point & m is mass per unit length of the string

Soln of the eqn (1) is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time 't'

(OR)

A string is stetched and fastened to two points at $x=0$ & $x=l$. Motion is started by displacing the string into the form $y = k(lx-x^2)$ from which it is released at time 0. find the displacement of any point on the string at a distance of 'x' from one end at time 't'

Soln: using the displacement $y(x,t)$ is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{—(1)}$$

$$y(0,t) = 0 \ \forall t \quad \text{—(2)}$$

$$y(l,t) = 0 \ \forall t \quad \text{—(3)}$$

$$y(x,0) = 0 \ \forall \ 0 \le x \le l \quad \text{—(4)}$$

$$\& \left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x) \text{ for } 0 \le x \le l$$

G.S for (1) using (2) & (3) we have

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

using condition (4) $\sum C_n \sin\frac{n\pi x}{l} = 0, \ 0 \le x \le l$

Hence $C_n = 0 \ \forall n$

$$D_n = \frac{2}{n\pi c}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{2\lambda}{n\pi c}\left[x(l-x)\cdot\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}} + (-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$\frac{2\lambda}{n\pi c}\left[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2\lambda}{n\pi l}\cdot\frac{2l^3}{n^3\pi^3}(1-\cos n\pi) = \frac{4\lambda l^3}{n^4\pi^4 c}(1-\cos n\pi).$$

If n is even $D_n = 0$

If n is odd $n = 2m+1$

$$D_{2m+1} = \frac{4\lambda l^3}{(2m+1)^4\pi^4 c}\cdot 2 = \frac{8\lambda l^3}{(2m+1)^4\pi^4 c}$$

$$y(x,t) = \frac{8\lambda l^3}{\pi^4 c}\sum_{m=0}^{\infty}\left\{\frac{\sin\frac{(2m+1)\pi ct}{l}\sin\frac{(2m+1)\pi x}{l}}{(2m+1)^4}\right.$$

(2) A tightly stretched string of length 'l' has its ends fastened at $x=0$, $x=l$. The mid-point of the string is then taken to height 'h' and then released from rest in that position find the lateral displacement of the point of the string at time 't' from the instant of release

Soln: Let $y(x,t)$ is displacement of the string

The initial displacement is given by OAB

Equation of OA is

$$y-0 = \frac{h-0}{l/2-0}(x-0)$$

$$\Rightarrow y = \frac{2h}{l}x$$

eqn of AB is

$$y-h = \frac{0-h}{l-l/2}\left(x-l/2\right) \Rightarrow y-h = (-h)\frac{2}{l}\left(x-l/2\right)$$

$$y = h-\frac{2h}{l}\left(x-l/2\right) = h\left[1-2/l\left(x-l/2\right)\right]$$

$$h\left[1-2/l\,x+1\right] = h\left(2-2/l\,x\right)$$

$$= 2h\left(1-x/l\right) = \frac{2h}{l}(l-x)$$

Thus the one-dimensional wave eqn

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}} \quad \text{--- (1)}$$

with boundary condition $y(0,t) = 0$, $y(l,t) = 0$ with

initial displacement $y(x,0) = f(x) = \begin{cases} \frac{2h}{l} x & \text{if } 0 \le x \le l/2 \\ \frac{2h}{l}(l-x) & \text{if } l/2 \le x \le l \end{cases}$

and $\left(\frac{dy}{dt}\right)_{t=0} = 0$

The soln of (1) satisfying the above boundary conditions and initial conditions is given by

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \quad \text{--- (2)}$$

where $A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$\frac{2}{l}\left[\int_0^{l/2} \frac{2h}{l} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} \frac{2h}{l}(l-x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$$

$$\frac{2}{l}\cdot\frac{2h}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^{l/2}$$

$$+\left\{(l-x)\frac{-\cos\frac{n\pi x}{l}}{n\pi/l} - (-1)\left[\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right]\right\}_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left\{-\frac{l}{n\pi} x \cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right\}_0^{l/2}$$

$$+\left\{\frac{-l}{n\pi}(l-x)\cos\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right\}_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$+\left\{0+0 + \frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$\frac{4h}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Sub the values of A in (2) we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$= \frac{8h}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{l}\right)\cos\frac{3\pi at}{l} + \cdots\right.$$

ONE DIMENSIONAL HEAT CONDUCTION EQUATION (OR) DIFFUSION EQUATION

The eqn of the form $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\cdot\frac{\partial u}{\partial t}$

(or) $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ — (1)

$c^2 = k/\rho s$, & c^2 is called the diffusivity the substance

The eqn (1) is called one dimensional heat flow eqn or diffusion eqn. Solution of (1) using method of separation of variables

problems:

(1) Find the temperature $u(x,t)$ in a bar OA of length 'l' which is perfectly insulated laterally & whose ends of A are kept at 0°C, given that the initial temperature at any point 'p' of the rod (where $op = x$) is given as $u(x,0) = f(x)$ $(0 \le x \le l)$

soln: The temperature distribution $u(x,t)$ is

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}} \quad — (1)$$

$u(0,t) = 0 \ \forall t$ — (2)

$u(l,t) = 0 \ \forall t$ — (3)

$u(x,0) = f(x)$ for $0 \le x \le l$ — (4)

The soln of the problem is

$$u(x,t) = (A\cos px + B\sin px)\,e^{-p^2c^2t}$$

using condition (2)

$$u(0,t) = 0 \Rightarrow A\,e^{-p^2c^2t} = 0 \ \forall t$$

$$\therefore A = 0$$

$$u(x,t) = B\sin px\, e^{-p^2c^2t}$$

using (3) condition

$$u(l,t) = 0$$

$$B.\sin pl.\,e^{-p^2c^2t} = 0$$

$$\Rightarrow \sin pl = 0$$

$$\Rightarrow pl = n\pi$$ where n is +ive integer

thus $p = n\pi/l$ where $n = 1, 2, 3 \ldots$

thus soln of (1) satisfying conditions (2) & (3)

$$u(x,t) = B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2}{l^2}.t} \quad \text{for } n = 1, 2, 3 \ldots$$

Hence if $u_1(x,t), u_2(x,t), u_3(x,t) \ldots$

are solns of (1) satisfying (2) & (3) conditions

The most general soln of (1) satisfying (2) & (3) is

$$\sum_{n=1}^{\infty} u_n(x,t)$$

$\therefore$ Hence the most general soln of (1)

satisfying conditions (2) & (3) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2}{l^2}.t} \quad \text{—(5)}$$

$\therefore$ B_n are arbitrary constants to be determined

using condition (4)

using condition (4) (i.e) $u(x,0) = f(x)$. putting $t = 0$ in (5)

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x), \quad 0 < x < l$$

$\therefore B_n = \frac{2}{L}\int_0^l f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,3\ldots$

(2) Solve the one dimensional heat flow eqn $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ given that $u(0,t)=0$, $u(L,t)=0$, $t>0$ & $u(x,0) = 3\sin\left(\frac{\pi x}{L}\right)$, $0<x<L$

Soln: Let $u(x,t) = X(x)T(t)$ be the soln of the eqn

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

given $u(0,t)=0$, $u(L,t)=0$, $t>0$ & $u(x,0) = 3\sin\frac{\pi x}{L}$, $0<x<L$

$u = XT$ put in (1)

$XT' = c^2 X''T$

$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda$ (say) where λ is a constant

$$X'' - \lambda X = 0 \quad \text{--- (2)}$$

$$T' - c^2\lambda T = 0 \quad \text{--- (3)}$$

(2) & (3) cases we have λ is +ive, zero (or) -ive

Case i: $\lambda>0$, let $\lambda = p^2$

then (2) & (3) becomes $X'' - p^2X = 0$

& $T' - c^2p^2T = 0$

solving these differential eqns, we get

$$X = A_1e^{px} + B_1e^{-px} \ \& \ T = C_1e^{p^2c^2t} \quad \text{--- (4)}$$

Case ii: Let $\lambda = 0$

Then (2) & (3) becomes $X''=0$ & $T'=0$

Solve these different we get

$$X = A_2 + B_2x \ \& \ T = C_2 \quad \text{--- (5)}$$

Comparing coefficients of different terms

$B_1 = 3,\ B_2 = B_3 = B_4 = 0$ ---(0)

Hence $u(x,t) = 3\sin\left(\frac{\pi x}{L}\right) e^{-\pi^2 c^2 t/L^2}$ which is required soln

(3) Derive the complete solution for the one dimensional heat eqn with zero boundary conditions problem with initial temperature $u(x,0) = x(L-x)$ in the interval $(0,L)$

Soln: The initial boundary value problem consists of

(i) P.D.E heat eqn $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(ii) zero Boundary conditions $u(0,t) = 0,\ u(L,t) = 0\ \forall t$

(iii) Initial condition $u(x,0) = x(L-x),\ 0<x<L$

Thus we have to find a temperature function $u(x,t)$ satisfying the differential eqn (i) subject to the boundary conditions (ii) and the initial condition (iii)

Now the soln of (i) is the form

$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-c^2 p^2 t}$ —(1)

By $u(0,t) = 0$ we have

$0 = c_1 e^{-c^2 p^2 t}\ \forall t$

$\Rightarrow c_1 = 0$

Now (1) reduced to

$u(x,t) = c_2 \sin px\, e^{-c^2 p^2 t}$

By $u(L,t) = 0$ we have

$0 = c_2 \sin pL\, e^{-c^2 p^2 t}\ \forall t$

$\sin pL = 0\ (\because c_2 \neq 0)$

$\Rightarrow pL = n\pi$, (or) $p = \frac{n\pi}{L}$ when n is any integer

Hence ② reduces to (7)

$$u(x,t) = b_n \sin\frac{n\pi x}{L}\, e^{-c^2 n^2\pi^2 t/L^2}$$

where $b_n = c_n$

Adding all such solns

The General soln of (1) satisfying the boundary condition (ii) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-c^2 n^2\pi^2 t/L^2}$$

put $t=0$, $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

In order that the initial condition (iii) may be satisfied (iii) and ④ must be same & this requires the expansion of $x(L-x)$ as a half-range fourier sine series in $(0,L)$ thus

$$x(L-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where } b_n = \frac{2}{L}\int_0^L (Lx - x^2)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{2}{L}\left[(Lx - x^2)\cdot\frac{-\cos\frac{n\pi x}{L}}{n\pi/L} - (L-2x)\frac{-\sin\frac{n\pi x}{L}}{n^2\pi^2/L^2} + (-2)\frac{\cos\frac{n\pi x}{L}}{n^3\pi^3/L^3}\right]$$

$$= \frac{2}{L}\left\{0+0-\frac{2L^3}{n^3\pi^3}\cos(n\pi)\right\} - \left\{0+0-\frac{2L^3}{n^3\pi^3}\right\}$$

$$= \frac{2}{L}\cdot\frac{2L^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$x(L-x) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8L^2}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

Hence ③ gives

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right) e^{-c^2 n^2\pi^2 t/L^2}$$

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left(\frac{(2n-1)\pi x}{L}\right) e^{-(2n-1)^2\pi^2 c^2 t/L^2}$$