Unit I: Simple Stress and Strains, Strain Energy

Introduction to Mechanics of Materials

Strength of Materials or Mechanics of Materials is a branch of applied mechanics that deals with the behaviour of deformable solid bodies subjected to various types of loadings. While studying Engineering Mechanics it is assumed that all bodies are either rigid or point particles. In this course on Strength of Materials, the bodies are considered deformable and subjected to failure or breakage. The focus is more on the internal effects in a body due to externally applied loads. This helps in determining the safe loads on a structure and is essential in the safe design of all types of structures like airplanes, antennas, buildings, bridges, ships, automobiles, spacecrafts, etc. This course forms the foundation for most engineering disciplines.

Mechanical Properties of Engineering Materials

The mechanical properties of a material are those which affect the mechanical strength and ability of material to be engineered into a suitable shape or application. Some of the typical mechanical properties of a material are as follows.

Strength: The *strength of a material* is its ability to withstand an applied load without failure. Failure is the state of the material in which it is no longer able to bear the applied load.

Elasticity: The property of a material by the virtue of which it returns to its original shape and size after removal of the applied load is called elasticity. The materials which follow such behaviour are said to be elastic.

Plasticity: The property of a material by the virtue of which it undergoes permanent deformations, even after removal of the applied loads is known as plasticity. The materials which are not elastic are said to be plastic.

Ductility: Ductility is a property which allows the material to be deformed longitudinally to a reduced section under tensile stress. Ductility is often categorized by the ability of material to get stretched into a wire by pulling or drawing. This mechanical property is also an aspect of plasticity of material.

Brittleness: Brittleness means lack of ductility. A brittle material cannot be deformed longitudinally to a reduced section under tensile stress. It fails or breaks without significant deformation and without any warning. It is an undesirable property from structural engineering point of view.

Malleability: Malleability is property of the material which allows the material to get easily deformed into any shape under compressive stress. Malleability is often categorized by the ability of material to be formed in the form of a thin sheet by hammering or rolling. This mechanical property is an aspect of plasticity of material.

Toughness: Toughness is the ability of material to absorb energy and gets plastically deformed without fracturing. Its numerical value is determined by the amount of energy per unit volume. It unit is Joule/ m3. Value of tough ness of a material can be determines by stress-strain characteristics of material. For good toughness material should have good strength as well as ductility. For example: brittle materials, having good strength but limited ductility are not tough enough. Conversely, materials having good ductility but low strength are also not tough enough. Therefore, to be tough, material should be capable to withstand with both high stress and strain.

Hardness: Hardness is the ability of a material to resist indentation or surface abrasion. Hardness measures are categorized into scratch hardness, indentation hardness and rebound hardness.

Concept of Stress and Strain

Stress: There are no engineering materials which are perfectly rigid and hence when material is subjected to external loads, it undergoes deformation. While undergoing deformation the particles of the material exert a resisting force. When this resisting force becomes equal to the applied load, an equilibrium condition takes place and deformation stops. This internal resistance is called stress.

Internal resistance per unit area is called intensity of stress (σ). Its SI unit is N/m or Pascal (Pa). It is common in engineering practices to specify the units of stress in N/mm^2 or MPa.

Consider a uniform cross-section bar under an axial load P. Let us pass an imaginary plane perpendicular to the bar along the middle so that the bar is divided into two halves. What holds one part of the bar with the other part is the internal molecular forces, which arise due to the external load P. In other words, due to the external load there is an internal resistive force that is generated which the holds the body together. This internal resistive force per unit area is defined as stress. If A is the area of cross-section of the bar, then the average stress (σ) on a given cross-sectional area (A) of a material, which is subjected to load P, is given by

$$\sigma = \frac{P}{A} \qquad (1)$$

Saint Venant's principle:

We must note that the above expression for stress is the average value of the stress over the entire surface. In reality, the stress varies along the cross-section, particularly at the ends of the bar carrying axial load, as shown in Figure 1.

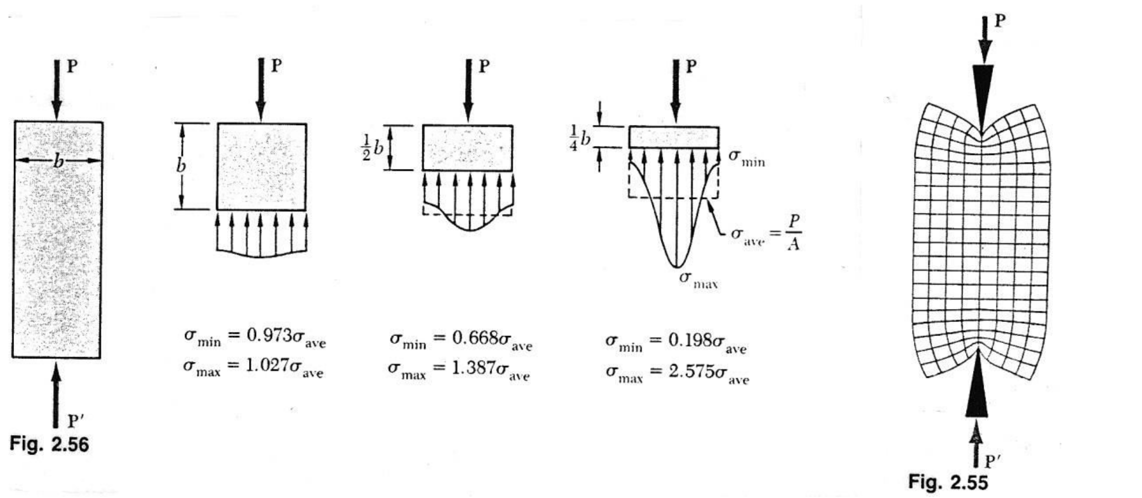


Figure 1. Illustration of Stress Concentration and St. Venant's Principle

(Image taken from Chapter 2, "Mechanics of Materials" by F.D. Beer, E.R. Johnston and J.T. DeWolf)

The stresses are highly concentrated in the immediate vicinity of the point of application of the load and reduce is magnitude as we move away from it along the cross-section. However, as we move away from the end of the bar towards the middle portion of the bar, the stress distribution becomes more uniform throughout the cross-section. Thus, away from the ends, the cross-sections can be assumed to have uniform stress, as given in equation (1). This is called the St. Venant's principle, which is more formally stated as:

The stresses in a deformable solid body at a point sufficiently remote from the point of application of the load depend only on the static resultant of the loads and not on the local distribution of the loads.

Figures (1) and (2) depict this principle.

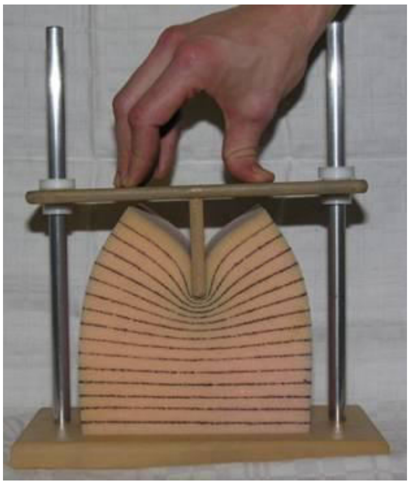


Figure 2. Illustration of St. Venant's Principle

Image taken from the website University of Manchester, U.K.:
http://www.mace.manchester.ac.uk/project/teaching/civil/structuralconcepts/Statics/stress/stress_mod3.php

Types of Stress

A) Normal/Direct Stress

1. Tensile Stress

The stress induced in a body, when subjected to two equal and opposite pull, as a result of which there is an increase in length, is known as tensile stress. Tensile stress tends to elongate the body.

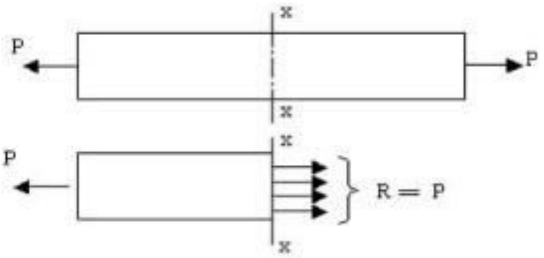


Figure 3. A bar subjected to tensile (axial) loading

Consider a uniform bar of cross-section area A subjected to an axial force P. The stress at any section, x-x, normal to the line of action of tensile force P is shown in the figure. The internal resistance R at x- x is equal to applied force P.

$$\text{Tensile Stress} = \frac{\text{Resisting Force (R)}}{\text{Cross Sectional Area (A)}} = \frac{P}{A} \qquad (2)$$

Under tensile stress, bar suffer stretching or elongation.

2. Compressive Stress

The stress induced in a body, when subjected to two equal and opposites pushes, as a result of which there is decrease in length, is known as compressive stress.

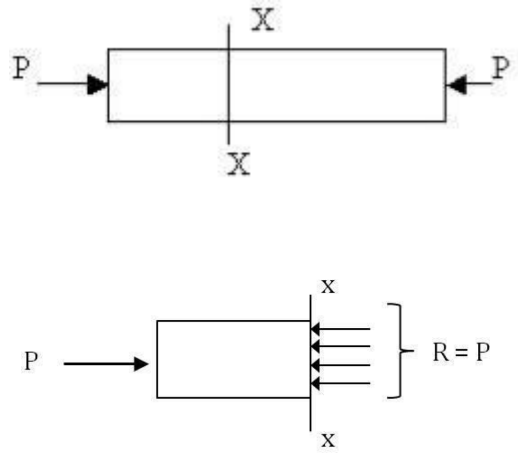


Figure 4. A bar subjected to compressive (axial) loading

Consider a uniform bar of cross-section area A subjected to an axial compressive load P. The stress at any section x-x normal to the line of action of compressive force P is shown in the figure. The internal resistance R at x-x is equal to applied load P.

$$\text{Compressive Stress} = \frac{\text{Resisting Force (R)}}{\text{Cross Sectional Area (A)}} = \frac{P}{A} \qquad (3)$$

Under compressive stress, bar suffers shortening.

Tensile Stresses are considered positive and compressive stresses are considered negative, as per general numerical sign convention for stresses.

B) Shear/Tangential Stress

A shear stress, symbolized by the Greek letter tau, τ, results when a member is subjected to a force that is parallel or tangent to the surface. The average shear stress in the member is obtained by dividing the magnitude of the resultant shear force V by the cross sectional area A. Shear stress is:

$$\text{Shear Stress, } \tau = \frac{\text{Shear Force (V)}}{\text{Cross Sectional Area (A)}} = \frac{V}{A} \qquad (4)$$

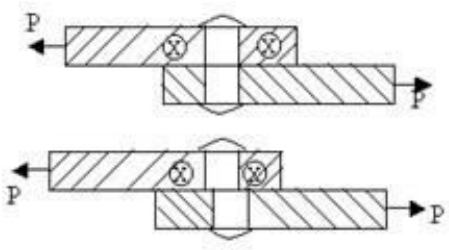


Figure 5. A rivet subjected to shear force

Consider a section of rivet is subjected to equal and opposite force P acting in a direction parallel to the resisting section. Such forces which are equal and opposite and act tangentially across the section, causing sliding of particles one over the other, are called shearing forces and corresponding stress induced in the rivet is called shearing stress.

Consider another example of a Clevis Joint as shown in Figure 6

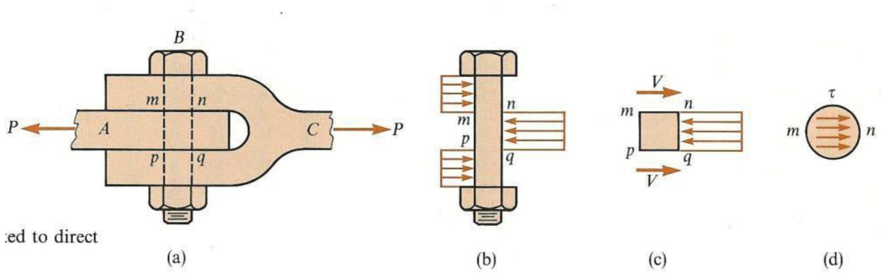


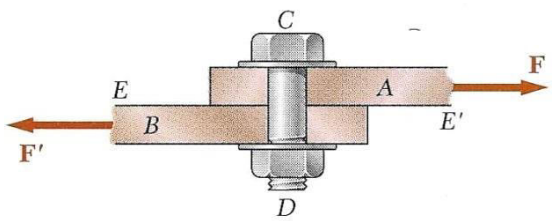
Figure 6. A rivet in a Clevis Joint subjected to shear

a) Typical clevis joint b) Free body diagram of bolt

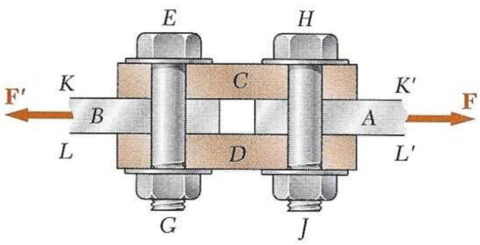
c) Free body of section mnqp d) Shear stresses on section mn

It should be noted that the distributions of shear stresses is not uniform across the cross section. Shear stress will be highest near the center of the section and become zero at the edge. This will be dealt in greater detail in Unit III.

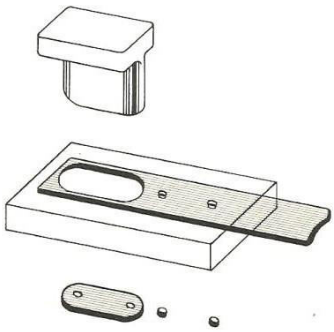
Direct or simple shear arises in the design of bolts, pins, rivets, keys, welds and glued joints.



(a) Single Shear Joint, Shear Stress = F/A



(b) Double Shear Joint, Shear Stress = F/2A



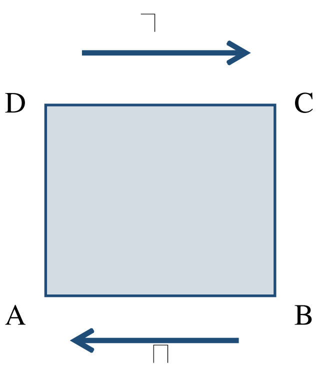
(c) Punching Shear = Punching force / Area

Figure 7. Examples of Single, Double and Punching Shear

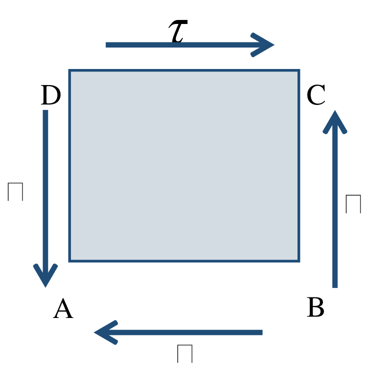
(Image taken from Chapter 1, “Mechanics of Materials” by F.D. Beer, E.R. Johnston and J.T. DeWolf)

Concept of complementary shear stresses

Consider an element ABCD from a material subjected to shearing stress (τ) on a faces AB and CD as shown in the Figure 8 (a), due to equal and opposite forces F applied onto the two faces. Since the element is in static equilibrium, it is not just the horizontal forces that are in balance, but the moment also has to be balanced. This unbalanced moment is balanced by counter couple on the two perpendicular faces BC and AD, as shown in figure 8 (b).



(a)



(b)

Figure 8. (a) Unbalanced Moments due to Shear

(b) Complementary Shear exists for Moment balance

To understand the existence of complementary shear, consider the following illustration. Suppose that a material block is divided into a number of rectangular elements, as shown by the full lines of Figure 9.

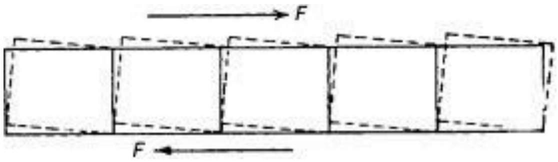


Figure 9. Illustration of existence of complimentary shear

(Image and concept taken from Chapter 3, “Strength of Materials and Structures (4th Edn.)”

by John Case, Lord Chilver and Carl Ross – Arnold Publishers, London)

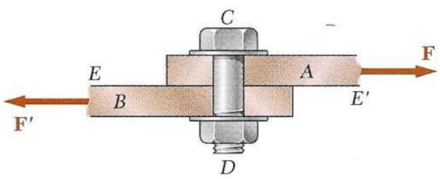
Under the actions of the shearing forces F, which together constitute a couple, the elements will tend to take up the positions shown by the dotted lines in Figure 9. It will be seen that there is a tendency for the vertical faces of the elements to slide over each other. Actually the ends of the elements do not slide over each other in this way, but the tendency to so do shows that the shearing stress in horizontal planes is accompanied by shearing stresses in vertical planes perpendicular to the applied shearing forces. This is true of all cases of shearing action a given shearing stress acting on one plane is always accompanied by a complementary shearing stress on planes at right angles to the plane on which the given stress acts.

C) Bearing Stress

A bearing stress, symbolized by the Greek letter sigma σ_b, is a compressive normal stress that occurs on the surface of contact between two interacting members. The average normal stress in the member is obtained by dividing the magnitude of the bearing force F by the area of interest. Bearing stress for the situation in Figure 10 is

$$\sigma_b = \frac{\text{Punching Force}}{\text{Contact Area}} = \frac{P}{A_b} = \frac{P}{td} \qquad (5)$$

Bolts, pins and rivets create bearing stresses along the surface of contact.



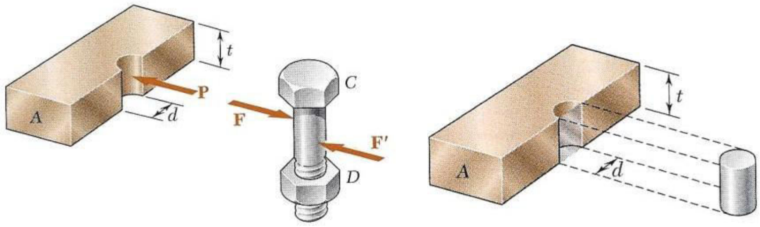


Figure 10. Demonstration of Bearing Stress

(Image taken from Chapter 1, "Mechanics of Materials" by F.D. Beer, E.R. Johnston and J.T. DeWolf)

Strain

Strain is a measure of deformation produced by the application of external force. It is the ratio of change in length to original length. It is denoted by Epsilon ( ). Strain is dimensionless. Strain in direction of applied load is known as linear or longitudinal strain.

$$\text{Strain}\,(\varepsilon) = \frac{\text{Change of length}\,(\delta l)}{\text{Original length}\,(l)} \tag{6}$$

Types of Strain

1. Tensile strain

Let initial length of bar before applied load be , when tensile load P is applied. Let the bar be elongated by .

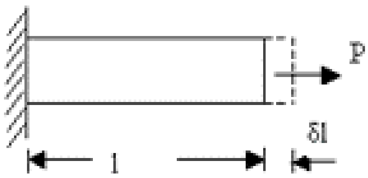


Figure 11. Tensile Strain

Tensile strain (ε) = ____________ (7)

2. Compressive strain

Let initial length of bar before applied load be , when compressive load P is applied. Its length gets decrease by .

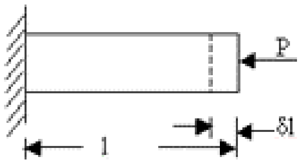


Figure 12. Compressive Strain

Compressive Strain (□)

$$\rule{3cm}{0.4pt} \qquad (8)$$

3. Shear Strain (□)

To get a proper definition of strain, it is important to understand the concept of complementary shear stresses in a material as discussed in the previous section.

Consider again the element ABCD from a material subjected to shearing stress (□) on a faces AB and CD as shown in Figure 13 (a). We may assume that the deformation occurs as shown in Figure 13 (b). However, this is possible only when the base AB is glued to the bottom. If the element ABCD is the portion of the material as shown in Figure 14 (a) subject to shear forces, then its deformation will be as shown in Figure 14 (b). This deformation is more common case of shear deformation in materials.

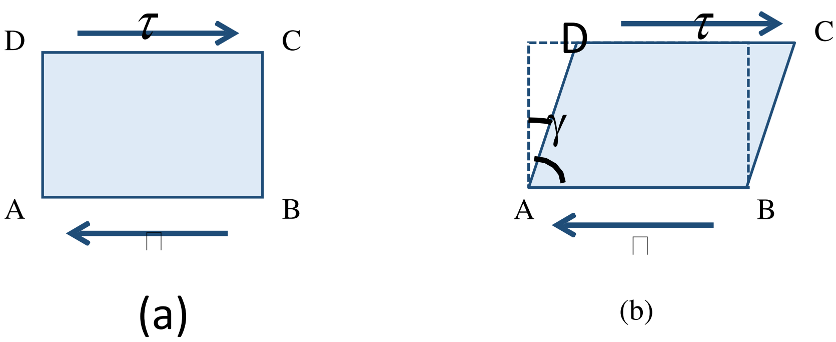
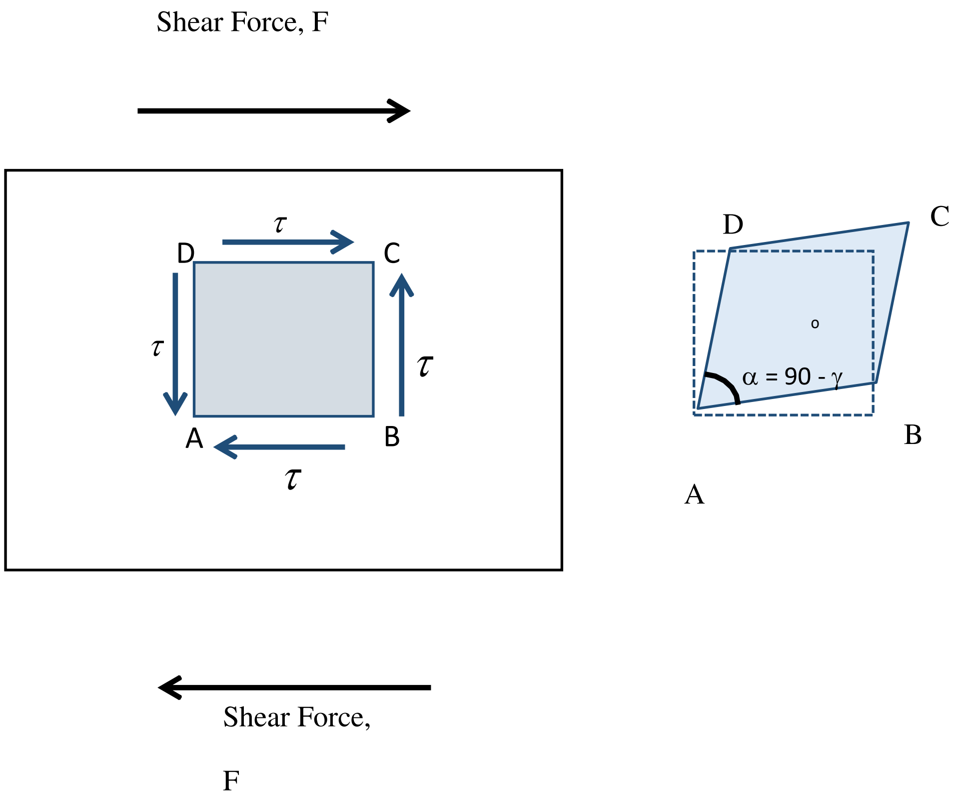


Figure 13. Shear deformation when edge AB is glued at the bottom



(a) (b)

Figure 14. Shear deformation in a plane in a general case

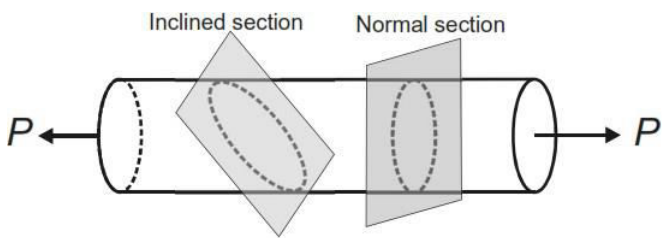
The shear strain is best defined as the change in the angle between the originally perpendicular edges of the rectangular element of the material, upon application of the shear stress. Original angle between AB and AD is 90^O. After deformation it is α. Thus, shear strain is defined as

$$\gamma = 90^O - \alpha \tag{9}$$

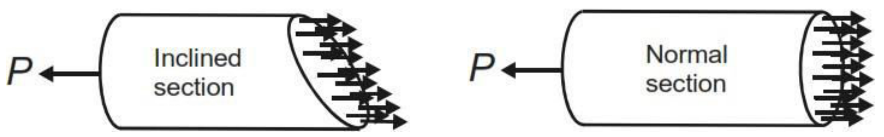
This definition is valid for both cases as shown in Figures 13 and 14.

Existence of Normal and Shear Stresses due to Axial Loading

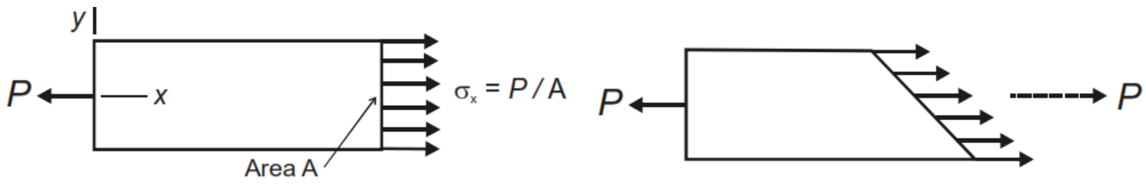
To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an "inclined" (as opposed to a "normal") section through the bar as shown in the Figure below. Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.



(a)



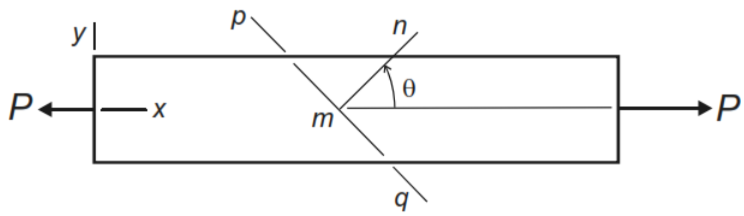
(b)



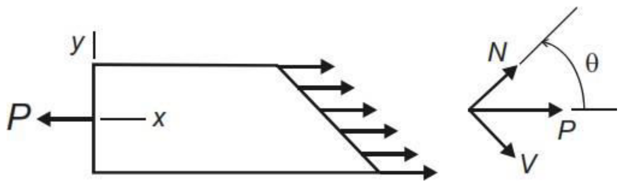
(c) Two-dimensional view of the previous image

Figure 15. Stress Distribution on an inclined section in axial loading

Specify the orientation of the inclined section *pq* by the angle θ between the x axis and the normal to the plane.



(a)



(b)

Figure 16. Resolution of the axial stress into normal and shear components

The force P can be resolved into components:

Normal force N perpendicular to the inclined plane, N = P cos θ

Shear force V tangential to the inclined plane V = P sin θ

If we know the areas on which the forces act, we can calculate the associated stresses. The area of the inclined plane section of the bar, *pq*, is A/ cos θ.

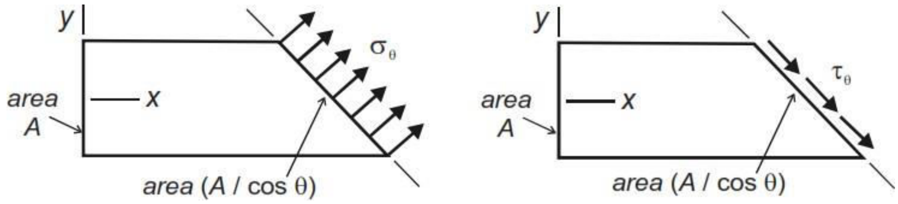


Figure 17. Normal and shear stresses on the inclined plane

Thus,

$$\sigma_\theta = \frac{Force}{Area} = \frac{N}{Area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta) \quad ☆ \qquad (10\text{ a})$$

$$\tau_\theta = \frac{Force}{Area} = \frac{-V}{Area} = \frac{-P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta) \quad ☆ \qquad (10\text{ b})$$

We can thus see that even a normal force offering axial load to a bar will produce both normal and shear stresses in the internal material of the bar. We may also note that when $\theta = 0^o$, that is when the plane pq is normal to the load P, we get normal stress as maximum and equal to P/A, while the shear stress on plane pq is zero. Another very important observation is that when $\theta = 45^o$, the shear stress is maximum and is equal to P/2A (in magnitude). The maximum shear stress produced is half the value of maximum normal stress.

Hooke's Law

Within elastic limit or more accurately up to the proportional limit of the material, the stress is directly proportional to strain.

For normal stress, the Hooke's law gives or

(11a)

Where = Axial/Normal Stress

= Axial/Normal Strain

E is known as Modulus of Elasticity or Young's Modulus or Elastic Modulus.

Young's Modulus (E) =

For shear stress, the Hooke's law gives □ □ or

□ □ (11b)

Where □ = Shear Stress

□ = Shear Strain

G is known as Modulus of Rigidity or Shear Modulus.

Modulus of Rigidity (G) = $\frac{}{□}$

Simple Tension Test for Mild Steel Specimen on Universal Testing Machine (UTM)

To study the behaviour of ductile materials in tension, a standard mild steel specimen is used for tensile test on universal testing machine (UTM).

- On the UTM more than one test can be performed like Tension, Compression, Bending and Shear etc.
- The end of specimens is gripped in UTM and one of the grips is moved apart by hydraulic jack or system, thus exerting tensile load on the specimen.
- The load applied is indicated on dial and the extension in the initial stages is measured by using an extensometer fixed on specimen itself and later stages by scale fixed on machine.
- Almost all machines are provided with an autographic recorder which is directly records the load vs deformation curve (or) stress vs strain curve.
- To fix the extensometer on specimen, two points are marked on a portion of specimen. The distance between these points over which the extension is marked is called gauge length.

The load Vs deformation curve is not unique, even for the specimen of the same material. As the geometry (either length or cross-sectional area) of the specimen changes, the load-deformation curve also changes.

On the other hand, the Stress vs Strain curve for a material is unique, irrespective of the geometric dimensions of the material specimen. Thus, for studying engineering properties of a material, Stress-Strain curve is commonly used.

The following is an example of a Stress-Strain curve for mild steel specimen.

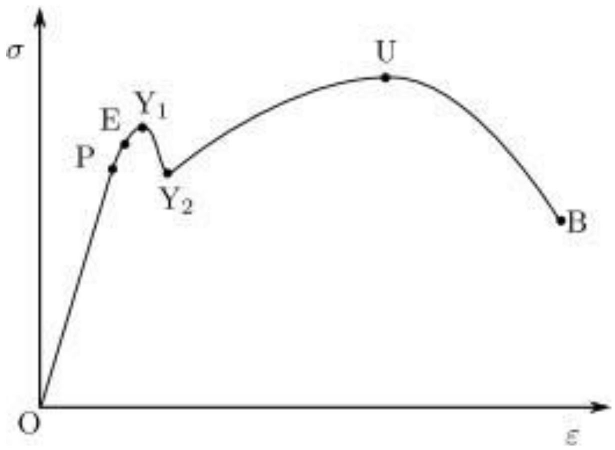


Figure 18. Stress-Strain curve for mild steel

Important Points on the Stress-Strain Curve

• P, Proportionality Limit:

The point up to which stress is linearly proportional to strain and hence Hooke's law is valid up to P. linear elasticity is valid. The slope of this line OP is nothing but the modulus of elasticity or Young's modulus.

• E, Elastic Limit:

The maximum stress that may be developed in a simple tension test such that there is no permanent or residual deformation when the load is entirely removed. Between P and E material is non-linearly elastic

• Y_1, upper yield point:

Point around which dislocations break through interstitial carbon atoms and relieve lateral strains. This phenomenon is particular to mild steels.

• Y_2 , lower yield point:

Once carbon atoms are overcome by the dislocation, relatively lower stresses are required to keep the dislocation moving. This happens around Y_2. This phenomenon is also specific to mild carbon steel. The stress corresponding to this point is called as yield stress or yield strength.

Note: Yield stress is also defined, in other words, as the stress at the which the material begins to deform plastically.

• U, Ultimate Stress

Maximum stress in a tensile test is reached at this point. The stress corresponding to this point is called as the Ultimate Strength.

• B, Breaking point or Rupture

Point at which specimen fails, breaking into two. The stress corresponding to this point is called as the breaking strength or the rupture strength.

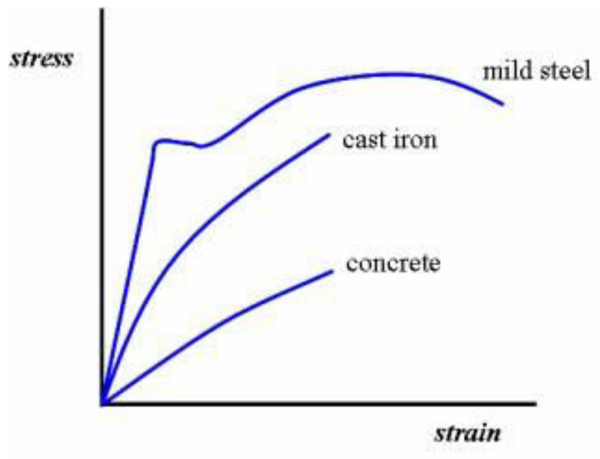
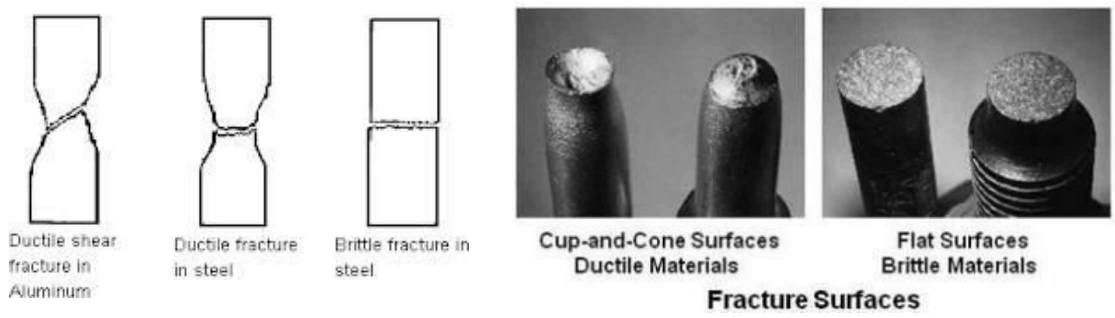


Figure 19. Stress-Strain curve for ductile and brittle materials

It should be notes that for ductile materials fail primarily due to shear stress, and their typical fracture mechanism is cup-come fracture mechanism. While the brittle materials fails, primarily due to normal stress and failure surface is flat, as shown in the following figures.



(a) (b)

Figure 20. Ductile and brittle failure patterns in materials

Concepts of Permanent Plastic Deformation (Slip, Creep) And Strain- Hardening

Cyclic Loading, Fatigue – Endurance Limit and Fatigue Limit Working Stress

and Factor of Safety

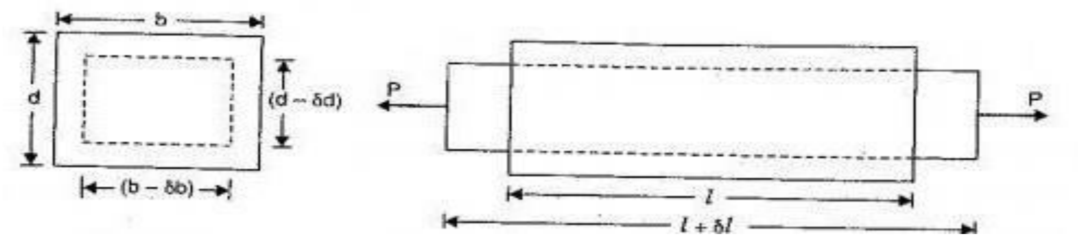
Working Stress: Maximum stress to which the material of a member is subjected in practice is called working stress. This is the stress to which a material is actually subjected to a stressed condition. To

avoid permanent setting in the member, working stress should be kept less than the elastic limit or permissible stress.
Factor of Safety: Ratio of yield stress to working stress is called factor of safety. And sometime factor of safety is taken as the ratio of ultimate stress to working stress.
It is necessary that the working stress should be well below the elastic limit and to achieve this condition, the ultimate stress is divided by factor of safety to obtain working stress.

Lateral Strain and Poisson's Ratio

Strain at right angles to the direction of applied load is known as lateral strain.

= Increase in length
= Decrease in breadth
= Decrease in depth



Lateral Strain = −

— or —

- If longitudinal strains tensile, lateral strain will be compressive.
- If longitudinal strain is compressive, lateral strain will be tensile.

Poisson's Ratio: Ratio of lateral strain to longitudinal strain is called Poisson's ratio. It is constant for given material, when the material is stresses within elastic limit. It is denoted by

or 1/m.

Poisson's ratio () = =

Poisson's Ratio varies from 0.25 to 0.33 for steel and 0.45 to 0.55 for rubber. The value of lies between 0 and 0.5.

Three Dimensional Stresses and Strains – Stress Tensor – Generalized

Hooke's Law

Volumetric Strain and Bulk Modulus

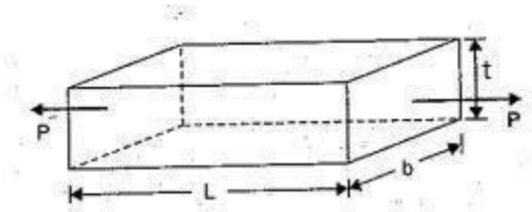
Volumetric Strain: Change in dimensions of body will cause some change in its volume. It is the ratio of change in volume to original volume.

= = + +

Consider a rectangular bar of length *l*, breadth *b* and thickness *t*, which is subjected to an axial load P in the direction of its length.

l = Original length of bar,
= Increase in length,
l+ = Final length,
b = Original breadth of bar,
= Decrease in breadth,
b − = Final breadth,
t = Original thickness of bar,
= Decrease in thickness,
t − = Final thickness
P = Tensile force acting on the bar, E = Modulus of elasticity
 = Poisson's Ratio

= —



Original Volume of body V = $l \times b \times t$

Final Volume = (l+) × (−) × (−)

Final Volume = l(1+)b(1−) t (1−)

Final Volume = lbt(1+)(1−) (1−)

Final Volume = lbt[{]

Final Volume = lbt [— — + — — — — — — — — —

—]

Ignoring other negligible value

Final Volume= ltb[— — —]

Change in Volume δV = Final Volume – Original Volume

$\delta V = lbt[1+ -] - lbt$

$\delta V = lbt[-]$

$$\frac{\delta V}{V} = [\text{ linear strain} - \text{lateral strain} - \text{lateral strain }]$$

$$\boldsymbol{\varepsilon_V = \frac{\delta V}{V} = [\text{ linear strain} - 2 \times \text{lateral strain}]}$$

$\delta V = V\,[\quad - \quad]$

Change in Volume $\delta V = lbt\,[\;\;-\;\;-\;\;]$

Change in Volume $\delta V = lbt\,[\;\;-\;\;-\;\;]$

Change in Volume $\delta V = V\,[1 - \;\; - \;\;]$

Change in Volume $\delta V = V$ [1 −2]

Volumetric Strain = = [1 −]

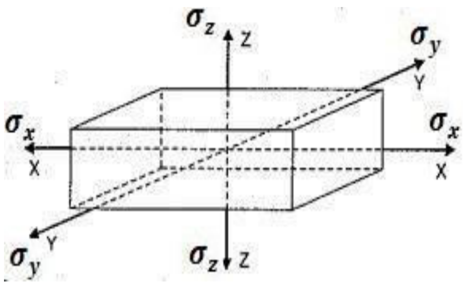
V

Volumetric Strain = [−]

Volumetric Strain $_V$ = [1 −

Volumetric Strain of a rectangular body subjected to three mutually perpendicular forces:

Let , , be the direct tensile stresses acting to x,y,z direction respectively



Volumetric Strain $_V$ = = $_x$ + $_y$ + $_z$

If intensity of stress is same in all direction i.e. = = = σ

Change in Volume δV = [1 −

Elastic Constant:

1) Modulus of elasticity: Ratio of longitudinal stress to longitudinal strain or linear stress to linear strain. It is denoted by E.

Modulus of elasticity (E) = $\frac{\text{}}{\text{}}$

1) Modulus of Rigidity: Ratio of shear stress to shear strain. It is denoted by G.

Modulus of Rigidity (G) = $\frac{\text{}}{\text{}}$

2) Bulk Modulus: When a body is subjected to three mutually perpendicular stresses of equal intensity, the ratio of direct stress to volumetric strain is known as bulk modulus. It is denoted by K.

Bulk Modulus K = ______ = ____

Relations between E, G and K

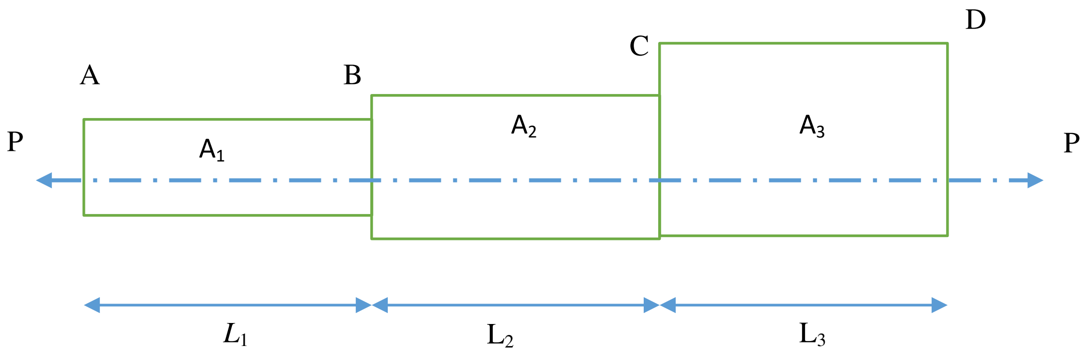
- E = 2G (1+) or E = 2G (1+μ)
- E = 3K (1 −) or E = 3K (1−μ)
- E = ____
- _ ____

Bars of Varying Sections

Figure shows a bar which consist of three lengths l_1 l_2 & l_3 with sectional area

A_1, A_2 & A_3 and subjected to an axial load P.

Even though the total force on each section is the same, the intensities of stress will be different for three sections.



Intensity of stress on Section AB = —

Intensity of stress on Section BC = $_2$ = —

Intensity of stress on Section CD = 3 = —

Now let E, be the Young's Modulus

Strain on Section AB = —

Strain on Section BC = —

Strain on Section CD = —

Change in length on Section AB = $_1$

= 1 1

Change in length on Section BC = $_2$

= 2 2

Change in length on Section CD = $_3$ = 3 $_3$

Total Change in length of bar = = 1 + $_2$ + $_3$

=+—

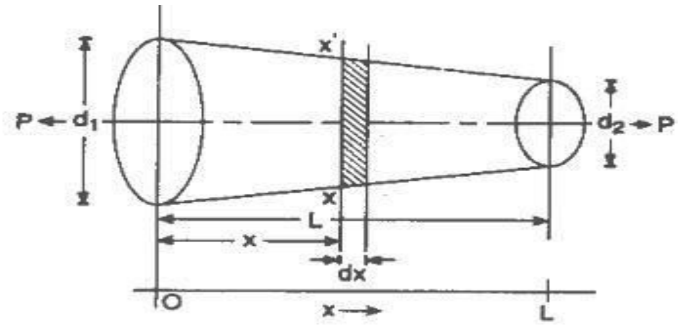
—= [+

Note: When a body is subjected to a number of forces acting on its outer edge as well as at some other sections, along the length of the body. In such case, the forces are split up and their effects are considered on individual sections. The resulting deformation of the body is equal to the algebraic sum of the deformations of the individual sections, called principle of superposition.

Stress in the bars of uniformly tapering circular sections

Consider a circular bar of uniformly tapering circular sections

Let P = Pull of bar, l = length of bar, d_1 = diameter of bigger end, d_2 = diameter of smaller end. Now consider a small element of length dx of the bar, at a distance x from the bigger end.



We can find out diameter of bar at a distance x from the left end A by using polynomial equation

D = a+bx ………………………………….(i)

Where D = diameter of taper section at a distance

x from left side In figure

At $x = 0$ at $x = l$

$D = d_1$ $D = d_2$

$a = d_1$ $d_2 = a + bl$ or $d_2 = d_1 + bl$ so b =

put in equation (i)

$D = d_1 + (\ )\ x$ or $D = d_1 + (d_2 - d_1)$

$D = d_1 - (d_1 - d_2)$ or D = Bigger end dia (major dia – minor dia)

$D = d_1 - kx$ Where k = ( )

Cross section are of the bar at this section A_x $= (d_1 - kx)^2$

Induced stress at this section

Strain at this section = = =

Elongation of elementary length = . dx =

.dx

Total extension of bar may be found out by integrating the above equation between the limit

0 to l.

= let $_1 - k = M$

$=^{-}(\ -\ )$

0 –k.dx = dM

Limit changing if $x = 0$ then $_1$= M

if x = l the M = $_1 - k$

=

= +

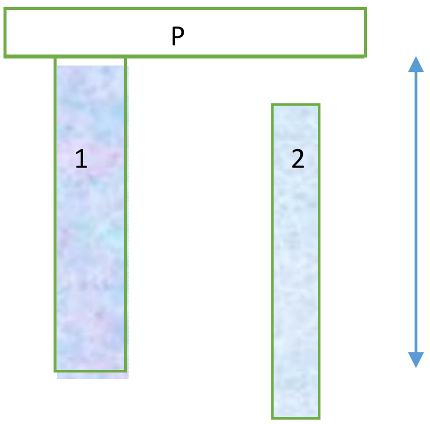
Put the value of k and by solving

Note: Same method & fundamental will apply for all tapering sections like square, Rectangular tapering section.

Bars of Composite section or Composite Bars

A bar made up of two or more than two different materials, joined together is called a composite bar.

1) The total external load on the composite bar is equal to sum of load carried by each different material
2) The extension or compression in each bar is equal. Hence deformation per unit length, i.e. strain in each bar is equal



P = Total load on composite bar

l = Length of composite bar and also length of bars of different materials A_1 = Cross-sectional area of bar 1

A_2 = Cross-sectional area of bar 2
E_1 = Young's Modulus of bar 1

E_2 = Young's Modulus of bar 2
P_1 = Load shared by bar 1

P_2 = Load shared by bar 2

$_1$ = Stress induced in bar 1

$_2$ = Stress induced in bar 2

ε1= Strain in bar 1

ε2= Strain in bar 2

By first point discussed above $P = P_1 + P_2$

$P = A_1 + A_2$

By second point discussed above that strain is same for both bar
Strain in bar 1 (ε1) = Strain in bar 2 (ε2)

— =— =—

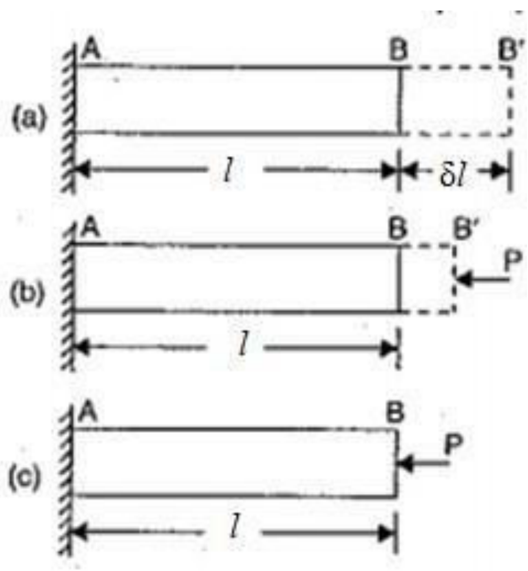
Where
— is known as Modular Ratio.

Thermal stresses or Temperature stresses

Thermal stresses are the stresses induced in a body due to change in temperature. Thermal stresses are setup in a body. When the temperature of body is raised or lowered and the body is not allowed to expand or contract freely. But if the body is allowed to expand or contract freely, no stress will be setup in the body.

You must have notice that in many times is structure will provide a gap between two structural elements. We allow the structural member to expand or contract due to variation in temperature. You must have noticed in the railway tracks are not continuous; some gaps are maintained at a certain distance travel. If this not, then rail track will be subjected to tremendous

amount of stress.



Consider a body which is heated at a certain temperature. l = original length of the body Δt = variation in temperature E = young's modulus

α = Co-efficient of linear expansion/ thermal expansion

δl = extension of rod due to rise of temperature

δl is proportional to strain $\Delta t.l$, Δt = + for expansion

$\Delta t = -$ for contraction

If the rod is free to expand $\delta l = \alpha\, \Delta t . l$

AB represents the original length and BB' represents the increase in length due to temperature rise, Now suppose that an external compressive load P is

applied at B' so that the rod is decreased in its length from (l+ $\alpha\, \Delta t.l$) to l = (l+ $\alpha\, \Delta t.l$) – δl = l

Total compressive strain

Thermal strain

= =

Thermal stress

= εT. = α E

If ends of body are fixed to rigid supports, so that its expansion is prevented then compressive stress and strain will be set up in the rod. These are known as thermal stress and thermal strain. If supports yield by an amount equal to d,

Then the actual expansion = expansion due to rise in temperature − δ

$$= \alpha\, \Delta t.l - \delta$$

Actual Strain = – ; Actual Stress = –

=

Strain Energy

Definitions

Strain Energy: When a material is deformed by external loading, energy is stored internally throughout its volume. Internal energy is also termed as strain energy.

Whenever a body is strained, energy is absorbed in the body due to straining effect is known as strain energy. The straining effect may be due to gradual applied load, sudden applied load or impact loading. The strain energy stored in the body is equal to work done by the applied load in stretching the body. Its S.I. unit is joule and 1 joule = 1 N-m

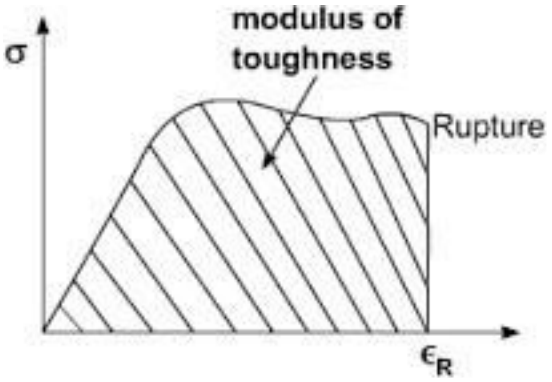
Resilience: The total strain energy stored in a body is known as resilience also defined as capacity of a strained body for doing work on the removal of straining force. A material resilience represents its ability to absorb energy without any permanent damage.

Proof Resilience: Maximum strain energy stored in a body is known as proof resilience. The strain energy stored in a body will be maximum when the body is stressed upto elastic limit. Proof resilience is the quantity of strain energy stored in a body when strained upto elastic limit.

Modulus of Resilience: The proof resilience per unit volume as a material is known as modulus of resilience.

Modulus of Resilience = ―――――――

Modulus of Toughness: Strain energy density of material before it fractures. It is used for designing member that may be accidently overloaded.



Expression for Strain Energy in a body when the load is gradually applied

Figure shows a load extension diagram of a body under tensile test upto elastic limit. The tensile load increase gradually from 0 to P and extension of body increase from 0 to δl Load P performs work in stretching the body. This work will be stored in the body as strain energy which is recoverable after the load P is removed.

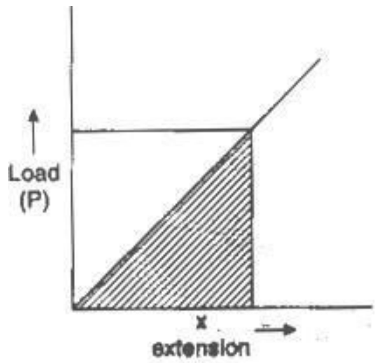
Let

P = Gradually applied load δl = Extension of body due to load A = Cross-sectional area of body

V = Volume of body l = Length of the body E = Young's Modulus σ

= Stress induced in the body

U = Strain energy stored in a body



Workdone by load = Area of load extension curve

= Area of triangle ONM = _

Or

Since P = σ × A = ε × l

× σ × ε × A

Workdone by load =_

_

_

$\varepsilon =$ $=$

Workdone by load $= -$

$-$

Workdone by load =

Modulus of resilience = Strain energy per unit volume

Modulus of Resilience =

Strain energy stored in a body when the load is suddenly applied

When we lower a body with the help of crane, the body is first of all, just above the platform on which it is to be placed. If the chain breaks at once at this moment the whole load of body begins to act on platform. This is the case of suddenly applied load. Let

P = Sudden applied load

δl = Extension of body due to load

A = Cross-sectional area of body

V = Volume of body

l = Length of the body, E = Young's Modulus
σ = Stress induced by sudden applied load U =
Strain energy stored in a body

Since the load is applied suddenly, therefore the load P is constant throughout the process of deformation of the bar.

Work done by load = Force × Distance

= Load × Deformation

= P× δl

We know that, Strain energy stored = Work done by load
Work done by load =

= P E = =

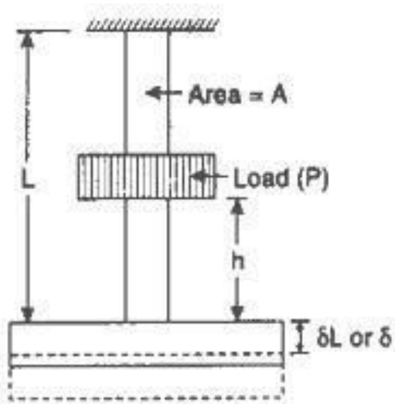
=

Work done by load =

$\sigma = 2$ _

It means stress induced in this case is twice the stressed induced when the same load is applied gradually.

Strain energy stored in a body when the impact load is applied Impact and shock load are almost same.



The load which is applied with some velocity on a body is known as impact or shock load. The kinetic energy of the load is absorbed as strain energy in the body.

This type of load is unsafe and must avoid in normal application.

Impact load are used in industry to produced forged parts and drive piles in the ground for reinforcing the earth for heavy building structure.

If a load is allowed to fall through a height, it will gain some velocity due to gravity and will strike the body with K.E.

Consider a vertical rod fixed at upper end and having a collar at the lower end. Let the load be applied from a height on a collar. Due to this impact load, there will be some extension in the rod.

P = Load applied with Impact

δl = Deformation of bar

A = Cross-sectional area of bar

h = Height through which the load will fall,

before impact on collar of bar V = Volume of bar l = Length of the bar E =

Young's Modulus

σ = Stress induced in rod due to impact load U
= Strain energy stored in a bar

Workdone by load = Force × Distance

= Load × Deformation

= P× (h+δl)

We know that, Strain energy stored = Workdone by load

= P. h + P.δl = P. h + P.

σ^2 ( ) - σ( ) - Ph = 0

Multiply both sides by

σ^2 ( ) - σ( ) - = 0

This is quadratic equation in σ

By roots formula

=

=

=

=

]

Once the stress obtained the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.

- When δl is very small as compared to h Workdone = P.h

= P.h

- If h = 0

  – $\sigma = \underline{\qquad}$

$\sigma = \underline{\quad}$

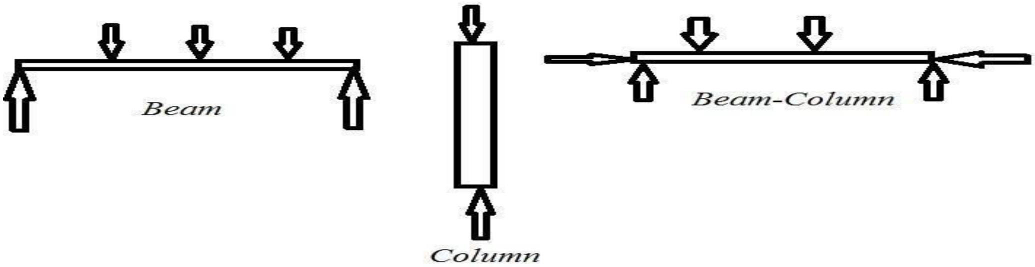
UNIT – II

Shear Force and Bending Moment

Definition of beam – Types of beams – Concept of shear force and bending moment – and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – Point of contraflexure – Relation between S.F., B.M and rate of loading at a section of a beam.

Definition: Beam

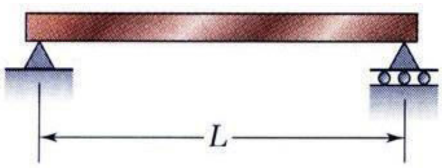
- A bar working under bending is generally termed as beam.
- A beam is laterally (Transverse) loaded member, whose cross-sectional dimensions are small as compared to its length.
- A beam may be defined as a structural member subjected to external loads at right angles to its longitudinal axis. If the external force acts along the longitudinal axis, it is called column. Material – Wood, Metal, Plastic, Concrete



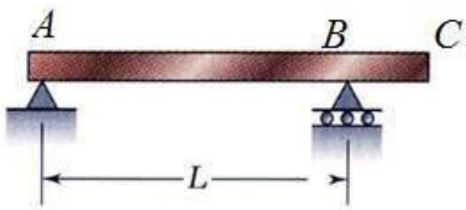
Types of beams: According to their support

1. ***Simply Supported beam***: if their supports creates only the translational constraints.

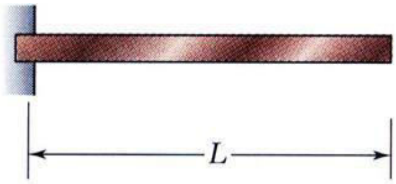
Sometime translational movement may be allowed in one direction with the help of rollers.



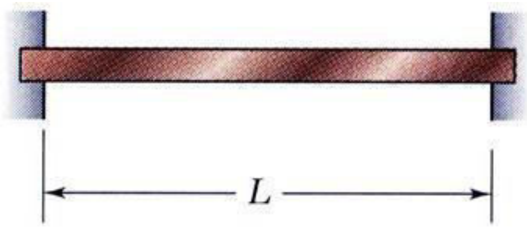
2. ***Overhanging beam***: A beam which is simply supported at point A and B and projects beyond point B. The segment BC is similar to cantilever beam but also the beam axis may rotate at point B.



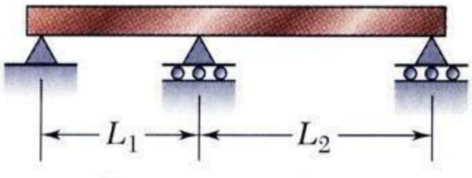
3. ***Cantilever beam:*** fixed at one end and free at other end. At fix support the beam can neither translate nor rotate, whereas at the free end it may do both. Therefore force & movement reactions may exist at the fixed support.



4. ***Fixed beam***: When both end is fixed.



5. ***Continuous beam***: More than two supports are there.



Types of loading

a. ***Concentrated or point load*****:** When external load acting on the beam is concentrated at a single point on the beam.
b. ***Uniformly distributed load (UDL):*** When external load acting on the beam is distributed over a length of beam, the following load is said to be a UDL. Ex- self-weight of beam, water pressure at bottom of water tank.
   It is represented as magnitude of load per unit length. For solving numerical problem, the total UDL is converted into point load, acting at the centre of UDL.

c. ***Uniformly varying load (UVL):*** Load spread over a beam in such a manner that rate of loading varies uniformly from point to point. Also known as triangular load.

Shear Force (SF)

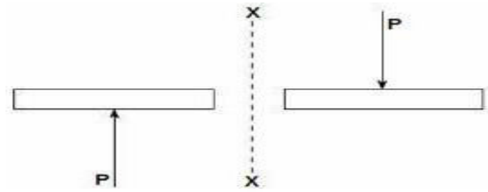
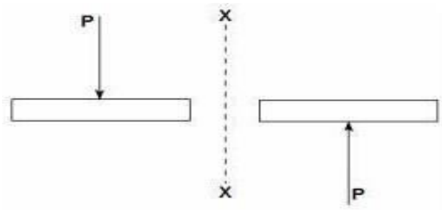
When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the term shear force and bending moment come into pictures.

The algebraic sum of vertical forces at any section of beam to the right or left of section is known as Shear Force.

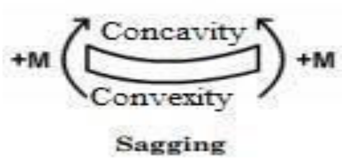
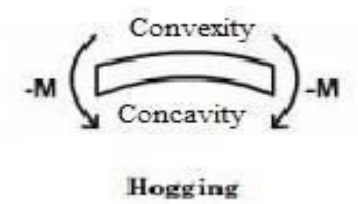
Bending Moment (BM)

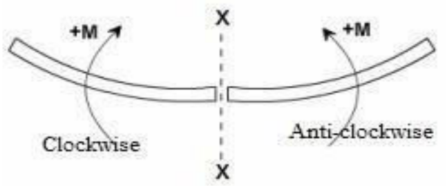
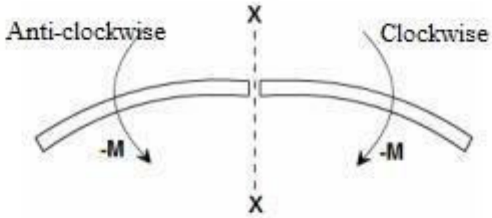
Algebraic sum of moments of all the forces acting to the right or left of the section is known as bending moment.

Sign Convention for Shear Force: Shear Force at section will be considered as

<table>
<tr><th>Positive (+)</th><th>Negative (-)</th></tr>
<tr><td>When resultant of forces to the left to the section is upwards or right of the section is downwards.

</td><td>When the left hand portion tend to slide downwards or right hand portion tend to slide upwards

</td></tr>
</table>

Sign Convention for Bending Moment:

<table>
<tr><th>Positive (+)</th><th>Negative (-)</th></tr>
<tr><td>We take bending moment as a section as positive, if it tend to bend the beam at that point to a curvature having concavity at top or convexity at the bottom.

</td><td>We take bending moment as a section as negative, if it tend to bend the beam at that point to a curvature having convexity at top or concavity at the bottom.

</td></tr>
<tr><td>Called Sagging Bending Moment</td><td>Called Hogging Bending Moment</td></tr>
</table>

When it is acting in clockwise direction to the left or anti-clockwise direction to the right. 	When it is acting in anti-clockwise direction to the left or clockwise direction to the right. 

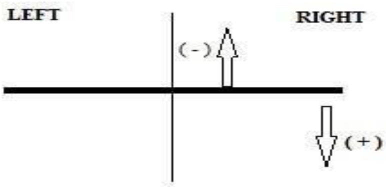
Shear Force Diagrams and Bending Moment Diagrams

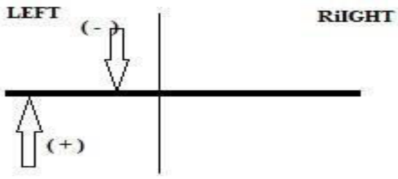
Before taking up the design of any structural element the structure is to be analysed and magnitude of bending moments and shear force should be determined. The structural element is designed for maximum bending moment and for maximum shear force. The BMD help to a great extent in identifying the tensile zones in reinforced concrete structure for providing steel reinforcement at appropriate place.

Important points for drawing SFD & BMD:

1. Consider a left or right portion of section.
2. Add forces (including reactions) normal to the beam on one of the portion.

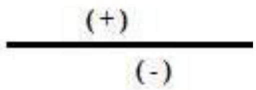
   If right portion of the section is chosen, a force on right portion acting downwards is positive, while force acting upwards is negative.



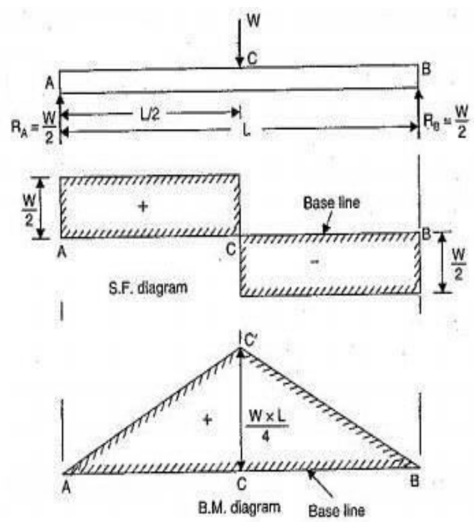


   If left portion of the section is chosen, a force on left portion acting upwards is positive, while force acting downwards is negative.

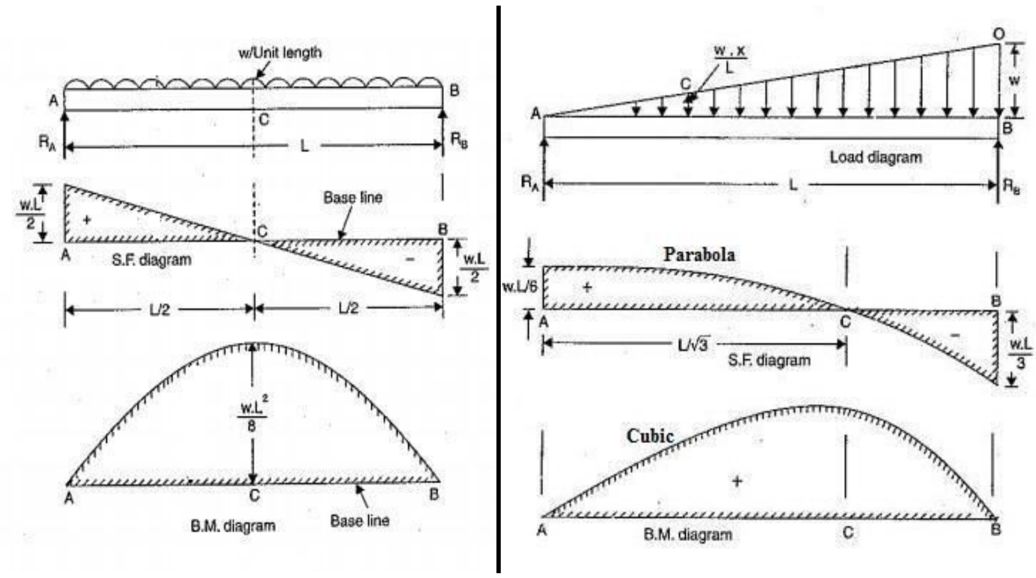
3. The positive value of SF and BM are plotted above the base line and negative value below the base line.



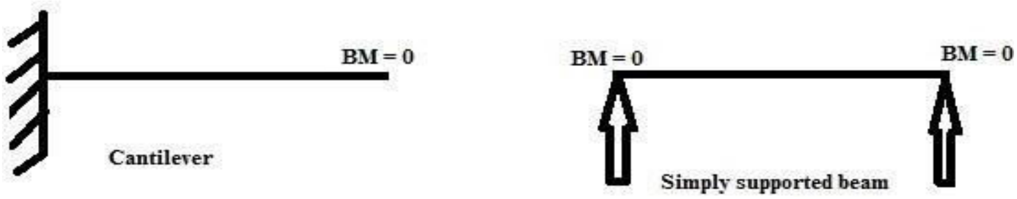
4. The SFD will increase or decrease suddenly i.e. by a vertical straight line at a section where there is a vertical point load but BM remains the same.
5. SF between any two vertical loads will be constant and hence SFD between two vertical loads will be horizontal.



6. If there is no load between two points, then the SF does not change (SF is horizontal) but BM changes linearly (inclined straight line).
7. If there is UDL between two points, then the SF changes linearly (SF is inclined by straight line) but BM changes according to parabolic law (BM will be parabola curve).



8. If there is UVL between two points, then the SF changes parabolic law (SF will be a parabola curve) but BM changes according to cubic law.
9. The BM at the two supports of a simply supported beam and the free end of cantilever will be zero.



10. Bending moment variation is 1 order a head than shear force variation.

Shear Force	**Bending Moment**
Rectangle or Constant	Linear or Triangle
Linear or Triangle	Parabola
Parabola	Cubic

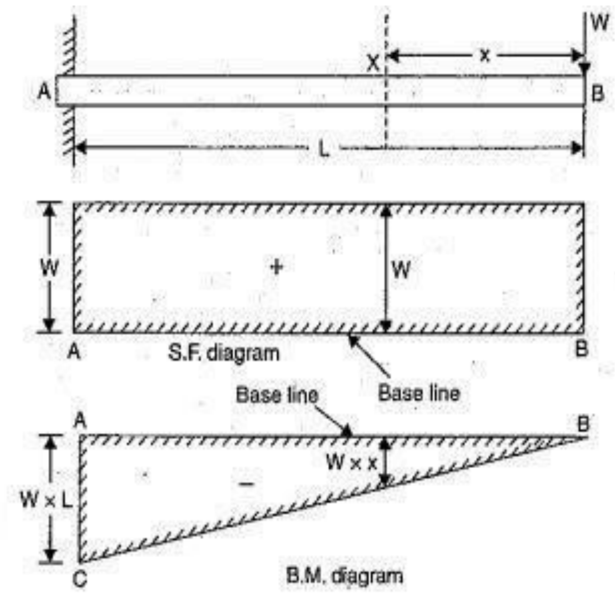
Common Relationships

Load	0	Constant	Linear
Shear	Constant	Linear	Parabolic
Moment	Linear	Parabolic	Cubic

Cantilever beam with a point load at its free end

Consider a cantilever beam AB of length *l*, fixed at A and free at B, and carrying a point load W at the free end B.

Calculation for SFD:



- Take a section X-X at a distance of x from free end B.
- Consider a right portion of the section.
- The shear force at this section is equal to resultant force acting on the right portion at given section.
- But the resultant force acting on the right portion at the section X-X is W and acting in downward direction.
- Force on right portion acting downward is considered positive.
- Hence shear force at section X-X is positive.
  SF at section X-$X = +W$
- There is no other load between A & B.
  So that Shear Force will be constant at all sections of cantilever beam.

Calculation for BMD:

Bending moment at section X-X = $M_x = -W.x$ ……………………………… *(i)*

- Bending moment will be negative as for the right portion of the section, the moment of W at X-X is clockwise.
- Bending of cantilever will take place in such a manner that convexity will be at the top of the beam.
- From equation (i) it is clear that BM of a cantilever beam at any section is proportional to the distance of the section from the free end
  BM at point $A_{(x=0)} = 0$

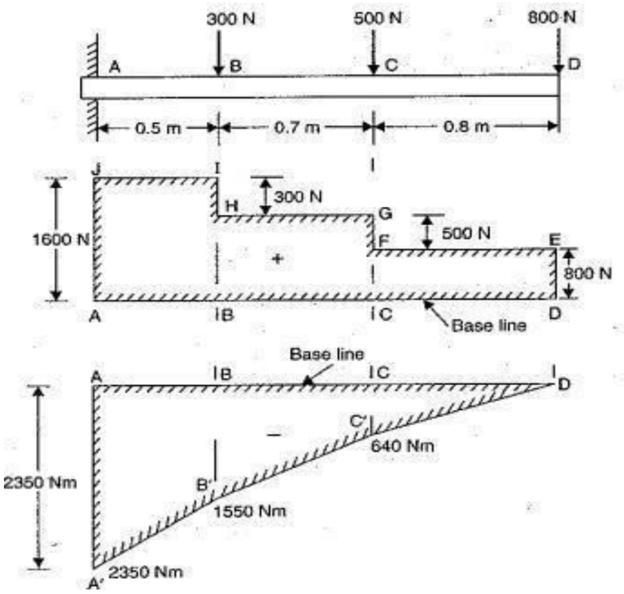
  BM at Point $B_{(x=l)} = -W.l$

- Hence bending moment follows straight line for such cases.

Q. A cantilever beam of length 2m carries the point loads as shown in figure. Draw shear force and bending moment diagram for cantilever beam.

Calculation for SFD:

SF at point D = 800N
SF just right to C = 800N
Sf at point C = 800+500 =
1300N SF just right to B =
1300N
SF at point B = 1300+300 = 1600N
Sf just right to A = 1600N

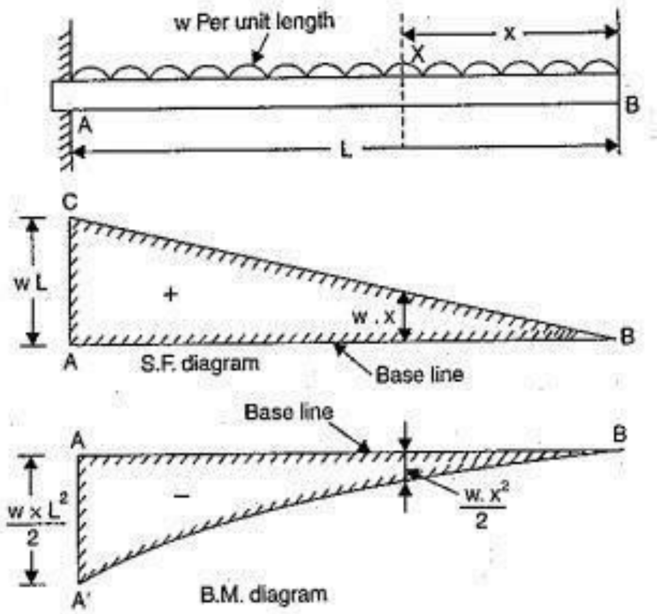


Calculation for BMD:

BM at point D = 0
BM at point C = 800×0.8 = 640N-m
BM at point B = 800×1.5 500×0.7
= 1550N-m
BM at point A = 800×2 500×1.2 300×0.5
= 2350N-m

Cantilever with Uniformly distributed load

Consider a beam AB of l length fixed at A and carrying a uniformly distributed load of w per unit length over the entire length of cantilever.



The SF at section *X-X* will be equal to the resultant force acting on the right portion of the section.
Resultant force on right portion = $w.x$

The resultant force is acting downwards.

Calculation for SFD:

SF at section *X-X* = *w.x*

SF at point $B_{(x=0)}$ = 0
SF at point $A_{(x=l)}$ =
w.l

Calculation for BMD:

As we discussed earlier that the UDL over a section is converted into point load acting at the C.G. of the section

BM at section *X-X* = . . =

BM at Point B*(x=0) = 0*

BM at point A*(x=l)* =

Q. A cantilever of length 2m carries a uniformly distributed load of 1kN/m run over a length of 1.5m from the free end. Draw the shear force and bending moment diagram for the cantilever.

Calculation for SFD:
SF at section *X-X* between B & C at a distance of x from free end = *w.x* = *x*
SF at point $B_{(x=0)}$ = 0

SF at point $C_{(x=1.5)}$ = 1.5kN
SF just right to A = 1.5kN

Calculation for BMD:
BM at section *X-X* between B & C at a distance of x from free end = w.x. =

BM at point $B_{(x=0)}$ = *0*

*B*M at point C*(x=1.5)* =

= −1.125kN-m
BM at section *Y-Y* between A & C at a distance of *y* from free end

= −1×1.5× [(*x* 1.5)] = 1.5 (*x* 0.75)

[Total load due to UDL is = 1×1.5 = 1.5kN

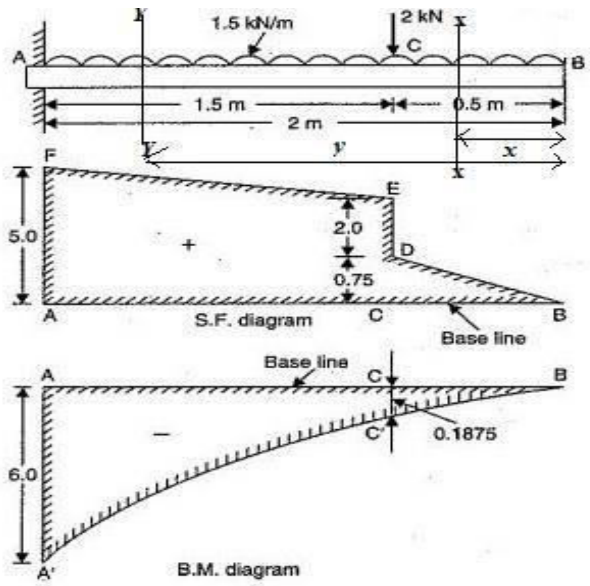
This load will act at a distance = 0.75m]

BM at point $C_{(x=1.5)}$ = 1.5 (1.5 0.75)

= 1.125kN-m
BM at point $A_{(x=2)}$ = 1.5 (2 0.75)
= 1.875kN-m

Q. A cantilever of length 2m carries a UDL of 1.5kN/m run over the whole length and a point load of 2kN at a distance of 0.5m from the free end. Draw SF and BM diagram.

Calculation for SFD:



SF at section *X-X* between B & C at a distance of x from free end = *w.x =1.5 x*

SF at point $B_{(x=0)} = 0$

SF just right to $C_{(x=0.5)} = 1.5\times0.5 = 0.75$kN

BM at any section *Y-Y* between A & C at a distance y from B (free end) is given by

BM at section *Y-Y* = (1.5×*y*+2*)* kN
SF at point $C_{(y=0.5)} = 0.75+2 =$
2.75kN

SF just right to $A_{(y=2)} = 5$kN

Calculation for BMD:
BM at section *X-X* between B & C at a distance of x from free end = −*w.x* .

2

= −1.5 = −0.75*x*

BM at point $B_{(x=0)} = 0$

*B*M at point $C_{(x=0.5)} = 0.75\times0.5^2$

= 0.1875kN-m

BM at section *Y-Y* between A & C at a distance of *y* from free end

= 1.5 . −2(*y-0.5)*

= *0.75* 2

2 (*y* 0.5) *B*M at point $C_{(y=0.5)}$

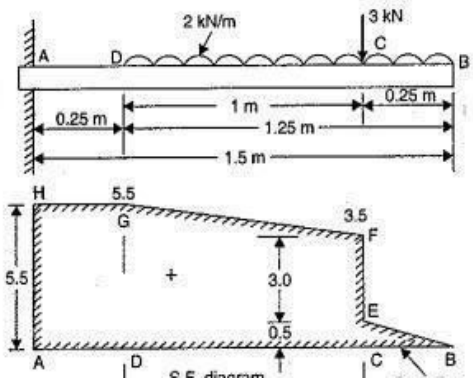
= 0.75×0.5^2
2 (*0.5* 0.5)

=
0.1875kN-m

BM at point $A_{(y=2)} = 0.75\times2^2$ 2 (*2* 0.5)

= 6kN-m

Q. A cantilever 1.5m long is loaded with a uniformly distributed load of 2kN/m run over a length of 1.25m from the free end. It also carries a point load of 3kN at a distance of 0.25m from the free end. Draw SFD & BMD of cantilever.



Calculation for SFD:

SF at point B = 0
SF just right to C = 2×0.25 =
0.50kN SF at point C = 0.50+3 =
3.50kN

SF just right to D = 3.5+2×1 =
5.5kN SF at point D = 5.5kN
SF just right to A= 5.5kN

Calculation for BMD:

BM at Point B = *0*

BM at point C = 2×0.25× ——

=

0.0625kN-m

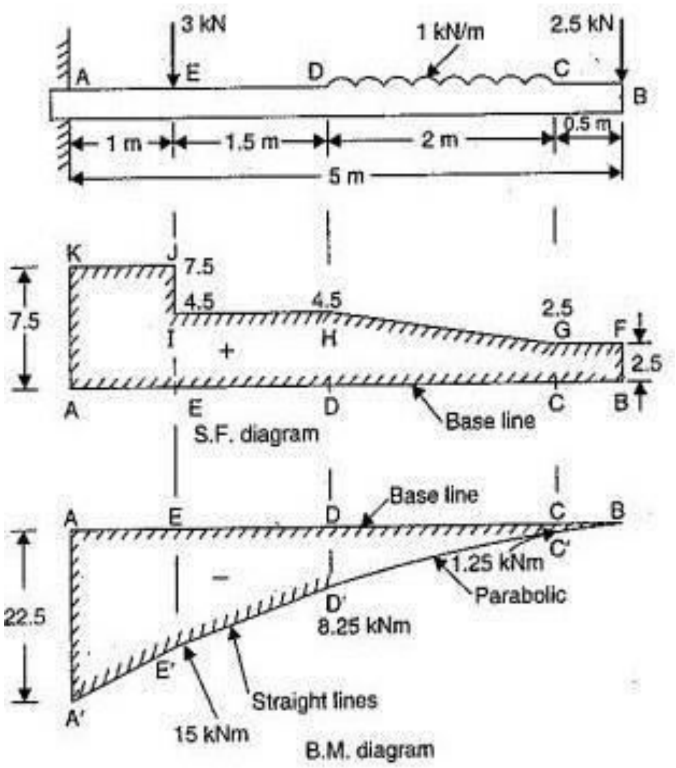
BM at point D =

2×1.25×=3×1

= 4.5625kN-m

BM at point A = 2×1.25×[—— + 0.25] −3 × [1+0.25] = 5.9375kN-m

Q. A cantilever of length of 5m is loaded as shown in figure. Draw SFD and BMD for cantilever.



Calculation for SFD:

SF at point B = 2.5kN

SF just right to C =

2.5kN SF at point C =

2.5kN

SF just right to D = 2.5+1×2 =

4.5kN SF at point D = 4.5kN

SF just right to E = 4.5kN

SF at point E= 4.5+3 =

7.5kN SF just right to A =

7.5kN

Calculation for BMD:

BM at Point B = 0

BM at point C = −2.5×0. 5 = −1.25kN-m

BM at point D = −2.5 × (2+0.5) −1×2×

= −8.25kN-m

BM at point E = −2.5× [0.5 + 2 +1.5] -1×2×[+1.5]

= 15kN-m

BM at point A = -2.5×5 – 1×2×[+2.5] −3×1

= 22.5kN-m

Cantilever with a gradually varying load

A cantilever of length l fixed at A and carried a gradually varying load from zero at free end to w per unit length at fixed end.

Rate of loading is 0 at B and is w per meter run at A

That means rate of loading for a length l is w per unit length

By using similar triangle ABE and CBD

That means =
=

CD
=
$\times$ x

Rate of loading for a length of x
= $\times$ x per unit length

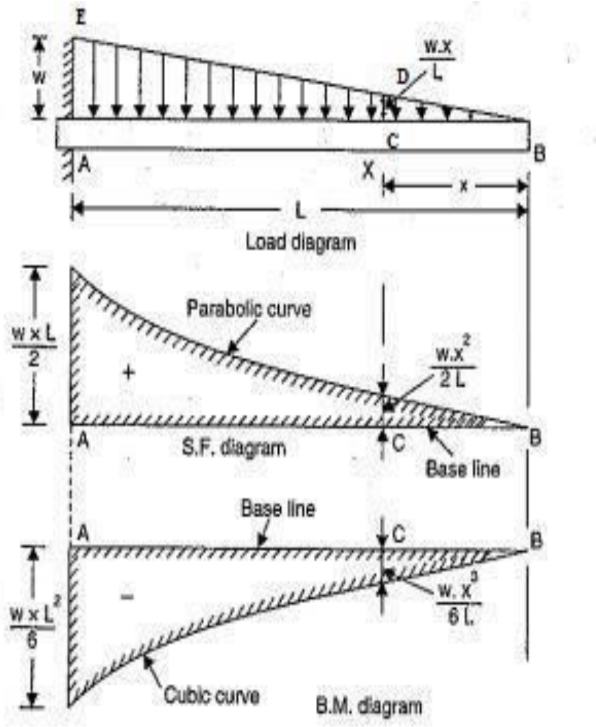
Calculation for SFD:

SF at section X-X between A & B at a distance of x from free end = Total load on the cantilever for a length of x from the free end B

= Area of triangle CBD CD = $\times$CB$\times$CD = $\times$ � $\times$ (
$\times$ x) = SF at point B(x=0) = 0

SF at point A(x=l) = =
=

Calculation for BMD:



BM at section X-X between A & B at
a distance of x from free end
= − (Total load for a length x) × Distance of load from C
= − (Area of Triangle CBD) × Distance of
C.G. of triangle from C

= − $\times$ −

= −

SF at point B(x=0) = 0

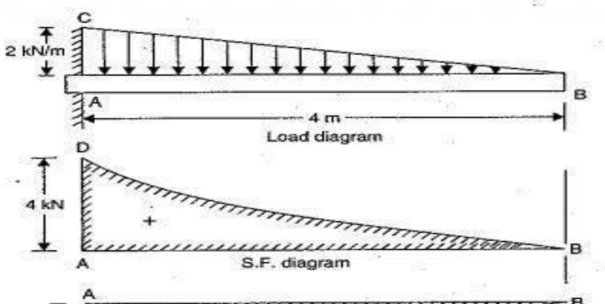
SF at point A(x=l) = = −

Q. A cantilever of length 4m carries a gradually varying load, 0 at free end to 2kN/m at the fixed end. Draw SFD and BMD for cantilever.

Calculation for SFD:
SF at point B = 0

SF just right to A = 4$\times$ = 4 kN

Calculation for BMD: BM at Point B = *0*

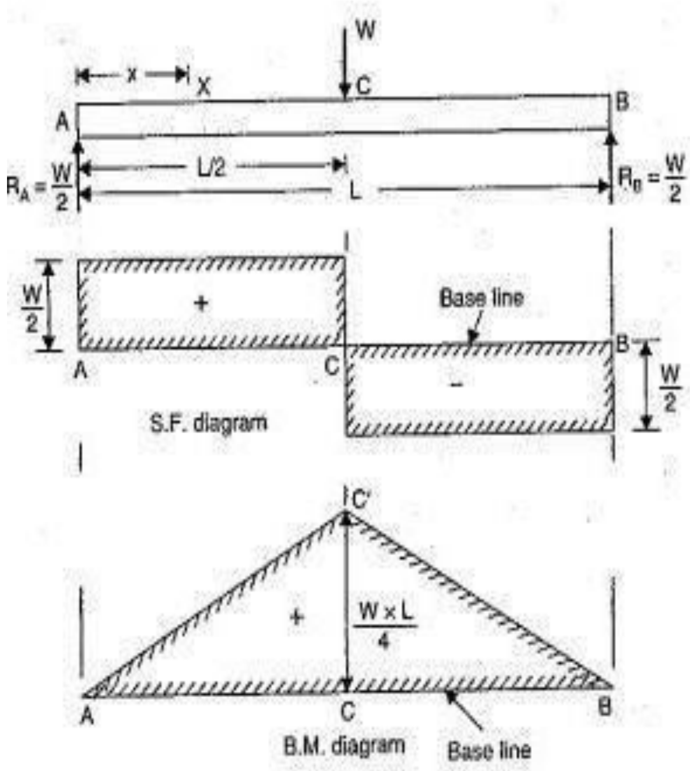


BM at point C =

—— = 5.33kN-m

Simply supported beam with a point load at its mid-point

A beam of length *l* simply supported at the ends A and B, carrying a point load W at its middle point C.



Calculation for Reaction:

RA +RB = W

Taking moment at point A, ΣMA = 0

RA×0+RB×l =—

RB =—

RA =—

Calculation for SFD:

SF at point A =—

SF at point C =— W = —

SF just left to B = –—

Calculation for BMD:

BM at section X-X between A & C at a distance of x from its left end = RA ×x—=

×x BM at point A(x=0) = 0

BM at point B(x=0) =— – —

BM at section Y-Y between B &C at a distance of y from its left end = R_A×Y – W(y–)

BM at point C(x=l/2) =—×l/2 = —

BM at point B(x=l) =—×l−W (l —) = — = 0 —

Simply supported beam with eccentric point load

Calculation for Reaction:

RA+RB = W

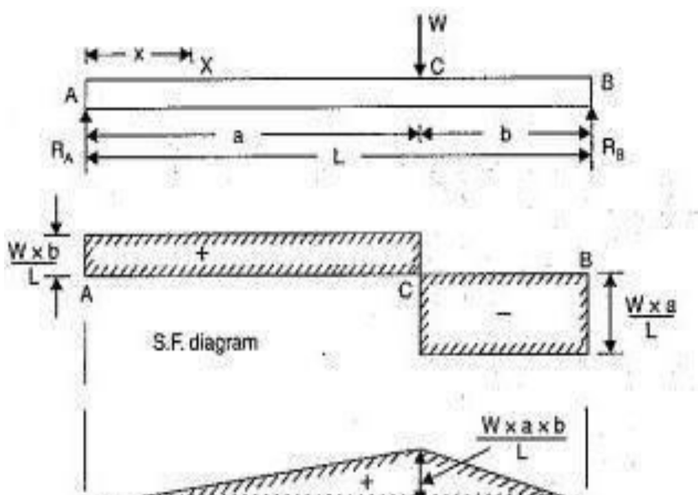
ΣMA=0

RA×0+l×RB =

Wa

RB =—

RA = W—= W (l−b)/l = —



Calculation for SFD:

SF at point A

= Sf at point ___

C = SF left to

B = –

___ – W = –___

___

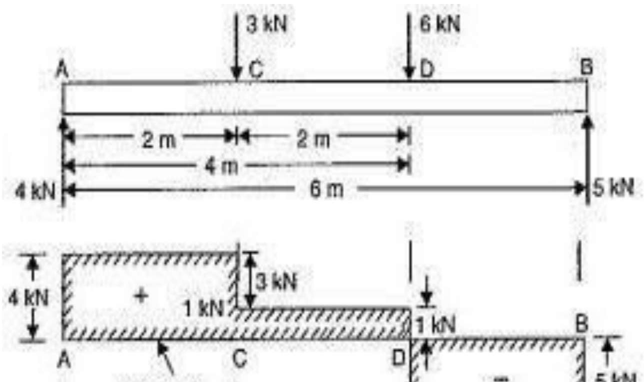
Calculation for BMD:

BM at point A = 0

BM at point C = ___

BM at point B = ___ – Wb = 0

Q. A simply supported beam of length 6m carries point load of 3kN and 6kN at distance of 2m and 4m from the left end. Draw SFD and BMD for the beam.



Calculation for Reaction:

RA + RB = 3+6

= 9 ΣMA = 0

RA×0 + RB×6 =

3×2+6×4 6RB = 30

RB =

5kN RA

= 4kN

Calculation for SFD:

SF at point A = 4kN

SF at point C = 4−3 = 1kN

SF at point D = 1−6 =
−5kN SF just left to B =
−5kN

Calculation for BMD:
BM at point A = 0
BM at point C = 4*2 = 8 kN-m
BM at point D = 4*4 – 3*2 = 10 kN-m
BM at point B = 4*6 - 3*4 – 6*2 = 0kN-m

Simply Supported beam with a uniformly distributed load

A beam AB of length l simply supported at the end A & B and carrying a uniformly distributed lod of w per unit length over the entire length.

The reaction at their support will be equal and their magnitude will be half the load on the entire length.

Calculation for Reaction:

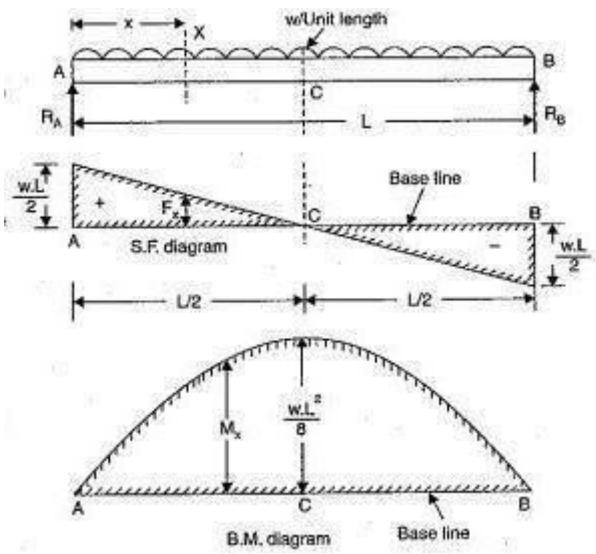
RA+RB =
w.l ΣMA=0

RA×0+l×RB = w.l.

RB=—

RA = — =
w.l− —

Calculation for Shear Force Diagram:



Shear Force at point A =—
Shear Force just left to C
=

— − w − = 0
×

Shear Force just left to B = —

Calculation for Bending Moment Diagram:

BM at point A = RA×0 = 0

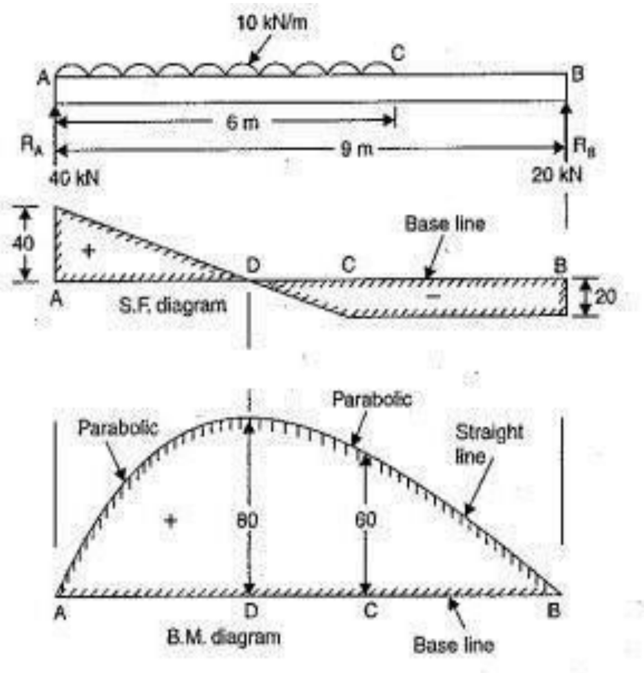
BM at point C = RA× w× × —

= ×— − —

= —

BM at point B = × w × × = 0

Q. Draw Shear Force Diagram and Bending Moment Diagram for a simply supported beam of length 9m and carrying a uniformly distributed load of a 10kN/m for a distance of 4m as shown in figure.



Calculation for Reaction:

RA+RB = 10×4 =
40 ΣMA=0

RA×0+9×RB = 10× 6−×

9×RB =
180 RB =
20 kN RA
= 40 kN

Calculation for Shear Force Diagram:

Shear Force at point A = 40 kN
Shear Force just left to C = 40 − 10 × 6
= −20 kN
Shear Force at Point C = −20 kN
Shear Force just left to B = −20
kN SF at A is +40 kN & at C =
−20 kN
SF between A to C varies by a straight line
This means somewhere between A to C, the SF is Zero
Let SF is Zero at a distance of x from A
Substitute SF value at this section equal to Zero
Shear Force at section X-X between A & C = 40 − 10×x = 0
x =
4m *Hence SF will be Zero at a distance of x= 4m*
from A ***Calculation for Bending Moment Diagram:***

BM at point A = RA×0 = 0

BM at point C = 40× 6 − 10 × 6−×

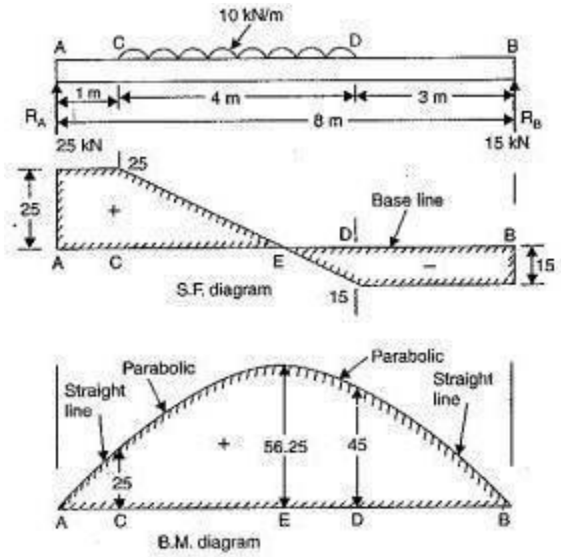
= 240 − 180 = 60 kN-m

BM at point B = 40 × 9 − 10 × 6 ×(+ 3) = 0

We know BM will be Maximum where SF changes its sign i.e. SF is Zero.

Maximum Bending Moment = 40× 4 − 10 × 4 ×= 80 kN-m

Q. Draw the Shear Force Diagram and Bending Moment Diagram for a simply supported beam of length 8m and carrying a uniformly distributed load of 10 kN/m for a distance of 4m as shown in figure.



Calculation for Reaction:

RA+RB = 10×4 =
40 ΣMA=0
RARA×0+8×RB = 10× 4× (
8×RB =
120 RB =
15 kN RA
= 25 kN

Calculation for Shear Force Diagram:

Shear Force at point A = 25
kN Shear Force at Point C =
25 kN
Shear Force at point D = 25 − 10 × 4 = -
15kN Shear Force just left to B = -15 kN
Shear Force at C = +25 kN and Shear Force at D = - 15 kN
That means somewhere between C & D, Shear Force is Zero
Shear Force at section X-X between C & D = 25 - 10× (� − 1) = 0
x = 3.5m
Shear Force at section X-X will be Zero.

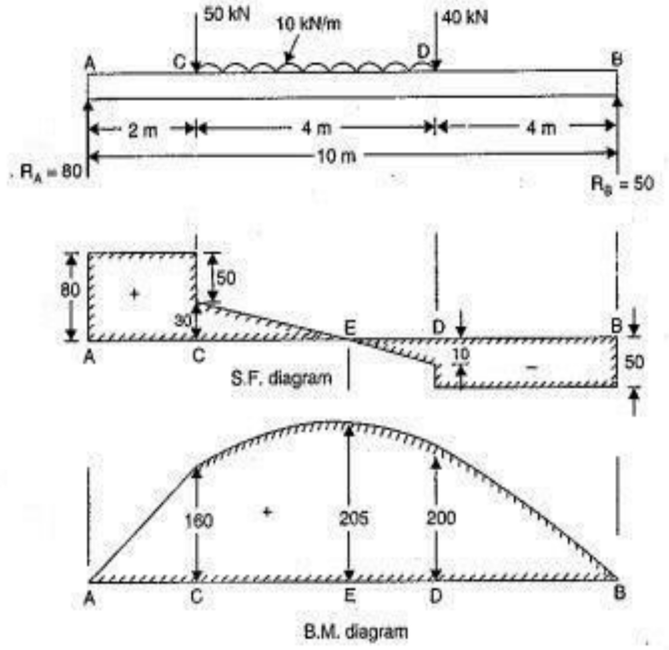
Calculation for Bending Moment Diagram:

BM at point A = RA×0 = 0
BM at point C =25× 1 = 25 kN-m
BM at point D= = 45 kN-m

BM at point B = 25 × 8 - 10 × 4 ×(+ 3) = 0 kN-m

Maximum Bending Moment (at x = 3.5m)-m =25 × 3.5 - 10 × 2.5 × = 56.25 kN-m

Q. A simply supported beam of length 10m carries the uniformly distributed load and two points loads as shown in figure. Draw Shear Force Diagram and Bending Moment Diagram for the beam. Also Calculate maximum bending moment of the beam.



Calculation for Reaction:

RA+RB = 50 + 40 + 10×4 = 130
kN ΣMA=0

RA×0+10×RB = 50 × 2 + 10× 4 × (+ 2)
+ 40 × 6
10×RB = 100 + 240
+160 RB = 50kN
RA = 80 kN

Calculation for Shear Force Diagram:

Shear Force at point A = 80 kN
Shear Force at Point C = 80 – 50 = 30 kN Shear
Force just left to D = 30 − 10 × 4
= −10 kN
Shear Force at point D = −10 − 40
= −50 kN
Shear Force just left to B = −50 kN
SF at section X-X between C & D = 80 – 50 − 10(x−2)
= 0 30 − 10(x−2) = 0� = 5m

Calculation for Bending Moment Diagram:

BM at point A = RA×0 = 0
BM at point C =80 × 2 = 160 kN-m
BM at point D = 80 × 6 − 50 × 4 − 10 × 4 × = 200 kN-m

BM at point B = 80 × 10 − 50 × 8 − 10 × 4 × (+ 4) − 40 × 4 = 0 kN-m

We know that Maximum bending moment will occur at that point where SF changes its sign
(SF = 0)

Maximum Bending Moment (at x = 5m) = 80 × 5 − 50 × 3 − 10 × 3 × = 205 kN-m

Simply supported beam carrying a uniformly varying load (Triangle load)
(from zero at both ends to w per unit length at the centre)

Consider a simply supported beam AB of span l and carrying a triangular load, varying gradually from 0 at both ends to w per unit length at the centre.

Since the load is symmetrical therefore the reaction R_A & R_B will be equal.

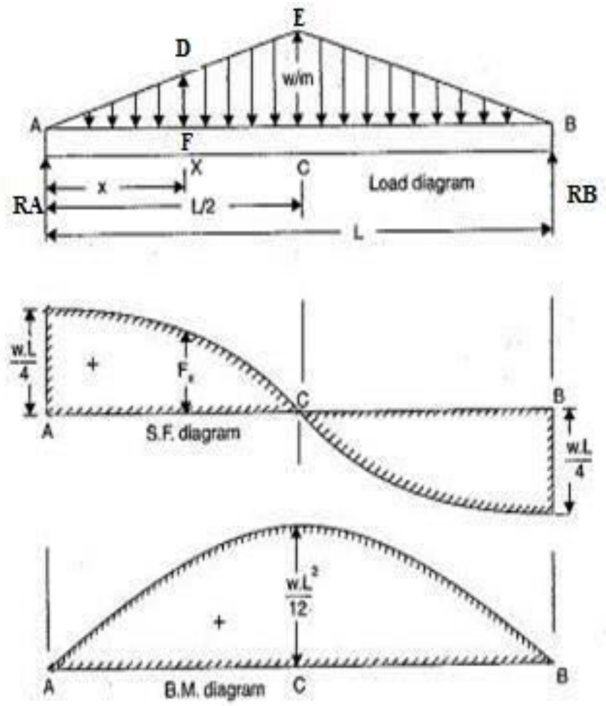
Total load on beam = Area of load diagram ABE

= ×× height = × × CE –

$= \frac{wl}{2}$

Hence $R_A = R_B$ = Half of the total load

R_A =— & R_B =—



Consider any section X-X between A & C at a distance x from A.

Rate of loading at section X-X =

DF ΔACE & Δ AFD is similar triangle, So

— — —— — ——

Load on length AF of the beam = Area of load diagram AFD

= ×× height = × F × DF–

$= \frac{1}{2} \times\ x \times \frac{2wx}{l}\ = \frac{wx^2}{l}$

This load will act at a distance of $\frac{x}{3}$ from DF

Calculation for SFD:

— ——

SF at section X-X = R_A – load on length AF

{It showing parabolic equation}

SF at point A$(x=0)$ =

SF at point C $= \quad - \quad = 0$

$(x= \;)$

SF just left to B$_{)}$ = − —

Calculation for BMD:

BM at section X-X = R_A× – load on length AF×

$$= \frac{Wl}{4} \times x - \frac{wx^2}{l} \times \frac{x}{3}$$

= — — {It showing cubic equation}

BM at point $A_{(x=0)} = 0$

BM at point C $= \times - \quad {}^3 = 0$

$(x= \;)$ × ( )

$$= \frac{wl^2}{8} - \frac{wl^2}{24} = \frac{wl^2}{12}$$

= — {where W = $\frac{Wl}{2}$ }

BM just left to B = 0

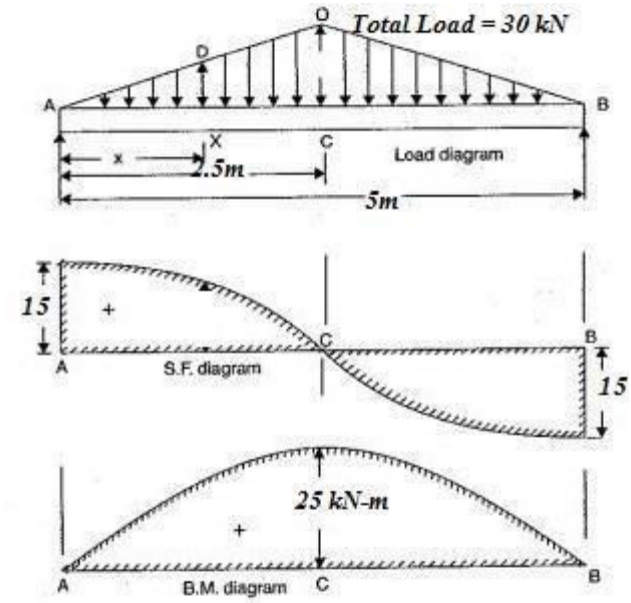
***NOTE** – Bending moment is always maximum, where Shear Force become zero after changing its sign*

***NOTE** – In case of simply supported beam, bending moment will always be zero at both ends (support)*

Q. A simply supported beam of 5m span carries a triangular load of 30kN, Draw SFD and BMD for the beam.

Calculation for reaction:

$R_A = R_B$ = Half of the total load $R_A = R_B = 15$ kN



Calculation for SFD:

SF at point A = 15 kN

SF at point C = 15 (30/2) = 0

SF just left to B = 15 kN

Calculation for BMD:

BM at point A = $R_A \times 0 = 0$

BM at point C =15×2.5 25 kN-m

BM at point B = 15×5 × 5 = 0kN-m

Simply Supported beam with a gradually varying load from zero at one end to w per unit length at other end

Consider a simply supported beam AB of length l and carrying a gradually varying load from 0 at A to w per unit length at B

Calculation for reaction:

Total load on beam = Area of triangle ABE

= × × height

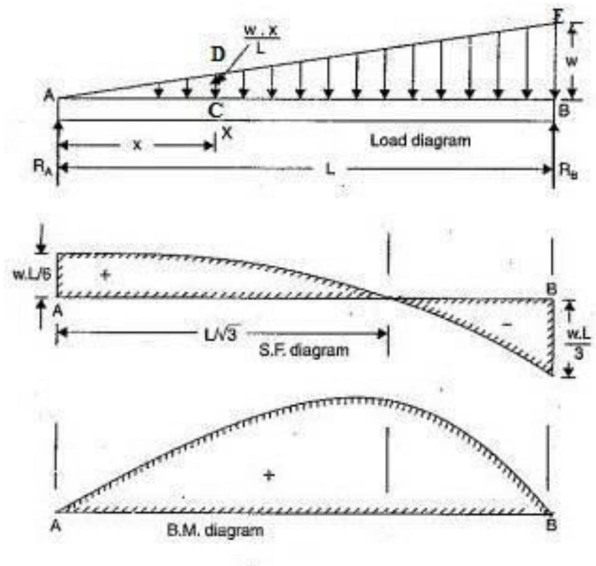
RA+RB = = $\times w \times l$

$\Sigma M_A=0$

RA×0+l×RB = ×

R_B =

R_A =



Consider any section X-X between A & B at a distance x from A

Rate of loading at section X-X = DC

ΔABE and ΔACD are two similar triangles

Calculation for SFD:

SF at section X-X = R_A – load on length AC

$$= \frac{Wl}{6} - \frac{1}{2} \times x \times \frac{wx}{l}$$ {It showing parabolic equation}

$$= \frac{Wl}{6} - \frac{wx^2}{2l}$$

SF at point A*(x=0)* =

SF just left to B = = –

We know that Maximum bending moment will occur at that point where SF changes its sign (SF = 0)

SF at section X-X between A & B at a distance of *x* from A $\frac{Wl}{6} - \frac{wx^2}{2l} = 0$ $\quad x = \frac{l}{\sqrt{3}}$

Or x = 0.577 l

Calculation for BMD:

BM at section X-X = R_A× – load on length AC×

$$= \frac{Wl}{6} \times x - \frac{1}{2} \times x \times \frac{wx}{l} \times \frac{x}{3}$$

$$= \frac{Wlx}{6} - \frac{wx^3}{6l}$$ {It showing cubic equation}

BM at point $A_{(x=0)} = 0$

BM at point C = – = 0

Calculation for Maximum Bending Moment:

BM will be maximum at that point where Shear Force is zero, i.e. at *x* =

Maximum bending moment $(\text{at} = \frac{l}{\sqrt{3}}) = \frac{Wl}{6} \times \frac{l}{\sqrt{3}} - \frac{w}{6l} \times \left[\frac{l}{\sqrt{3}}\right]^3 = \frac{wl^2}{9\sqrt{3}}$

Q. A simply supported beam of length 5m carries a uniformly increasing load of 800N/m run at one end to 1600N/m run at other end. Draw the shear force and bending moment diagram for the beam, also calculate the position & magnitude of maximum bending moment.

Weight may be assumed split into

a. Uniformly distributed load of 800N/m over the entire span

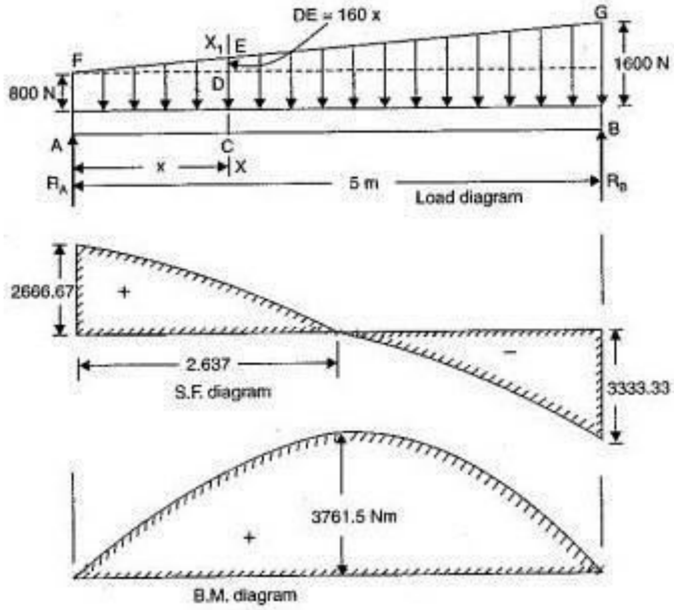
b. Uniformly gradually varying load of 0 at C & 800N/m at B

Total UDL = W_{UDL} = × l = 800 × 5 = 4000N

Total UVL = W_{UVL} = × × height = × 5 × 800 = 2000N

Total load acting on beam = W_{UDL} + W_{UVL} = 4000+2000 = 6000N

ΔCDE and ΔCGH are two similar triangles



$\frac{ED}{CD} = \frac{HG}{CC}$ $\frac{800}{5} = \frac{HG}{x}$

HG = 160

Calculation for reaction:

$R_A + R_B$ = 6000N ΣM_A=0

R_A×0+5×R_B = 800× 5 × − + 5 × (× 5 × 800) × ( )

$5R_B$ = 10000+6666.66

6×R_B = 54

R_B = 3333.33N

R_A = 2666.67N

Calculation for SFD:

SF at section X-X between A & B at a distance of x from A

= R_A – load on length AF

= R_A – (Area of rectangle + Area of ΔCGH)

$= 2666.67 - 800\times x - (\frac{1}{2} \times x \times 160x)$

= 2666.67 – 800 80 2

SF at point $A_{(x=0)}$ = 2666.67 N

SF at point B $_{(x=5)}$ =2666.67 – 800 80× 5^2 = 3333.33N

We know that, Shear Force changes its sign from positive to negative, that means at somewhere there will be a point where Shear will be Zero. And at this point Bending Moment will be maximum.

2666.67 – 800 80 2 = 0

= 2.637m

Calculation for BMD:
BM at section X-X between A & B at a distance of *x* from A

$$= R_A \times x - 800 \times x \times \frac{x}{2} - \left(\frac{1}{2} \times x \times 160x\right) \times \frac{x}{3}$$
$$= 2666.67\, x - 400x^2 - \frac{80x^3}{3}$$

BM at point A (x=0) = 0

BM at point B (x=5) = $2666.67 \times 5 - 400 \times 5^2 - \frac{}{}$

Maximum Bending Moment (x=2.637) = $2666.67 \times 2.637 - 400 \times 2.637^2 - \frac{}{}$

Maximum Bending Moment (x=2.637) = 3761.5 kN-m

Point of Contraflexure (or Point of Inflexion):

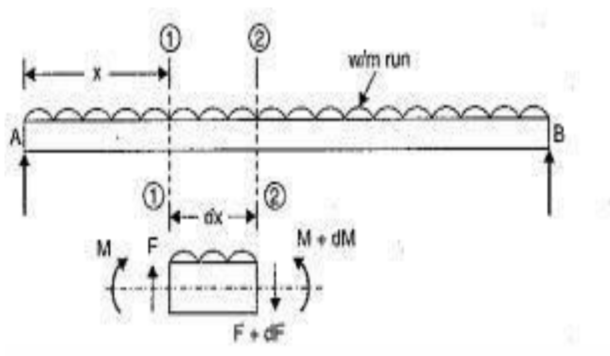
Bending moment in cantilever was negative, whereas that in a simply supported beam is positive. It is thus obvious that in an overhanging beam, there will be a point where Bending moment will change sign from negative to positive or vice-versa, such a point, where the bending moment changes sign is known as Point of contraflexure or point of inflexion.

- The point of contraflexure is the point where the bending moment changes its sign from positive to negative or bending moment from Sagging to Hogging and vice-versa.
- At point of contraflexure bending moment is zero.
- It is to be noted that all the points where BM is zero are not necessary point of contraflexure.

Shear Force and Bending Moment Diagram for overhanging beams:

If the end portion of the beam is extended beyond the support, such beam is known as overhanging beam. Overhanging can be assumed as a combination of simply supported beam and cantilever beam. Due to this, overhanging beam experienced point of contraflexure.

Relationship between Load, Shear Force and Bending Moment



Consider a simply supported beam carrying uniformly distributed load of *w* per unit length. An element *dx* at a distance of *x* from the left end A is considered.

An equilibrium of the portion of beam between sectoin1-1 and section 2-2 *F* = Shear Force at section 1-1

F + dF = Shear Force at section 2-2

M = Bending moment at section 1-1

M + dM = Bending moment at section 2-2

Total load on length dx of beam = *w.dx*

For equilibrium ΣV = 0

$F - w.dx = F + dF$

$- w.dx = dF$

$\frac{dF}{dx} = - w$

This shows that the rate of change of Shear Force is equal to load.

Taking moment at section 2-2 i.e. at point D

$M + F.dx - w.dx.dx/2 = M + dM$

Neglecting higher power of small quantities

$F.dx - dM = 0$

$F =$

This relationship shows that the rate of change of bending moment is equal to shear force.

– w =

– w = . =

w =

=

UNIT – III

FLEXURAL STRESSES, SHEAR STRESSES

Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/y = E/R - Neutral axis – Determination of bending stresses – Section modulus of rectangular and circular sections (Solid and Hollow), I,T, Angle and Channel sections – Design of simple beam sections.

Derivation of formula – Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T angle sections.

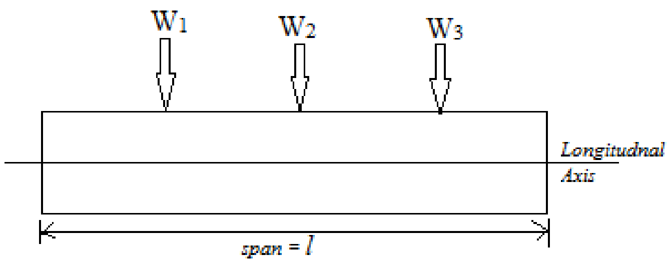
PART A: FLEXURAL STRESSES

Introduction

Beam: Beam is a structural member on which a system of external loads acts at right angles to its longitudinal axis.

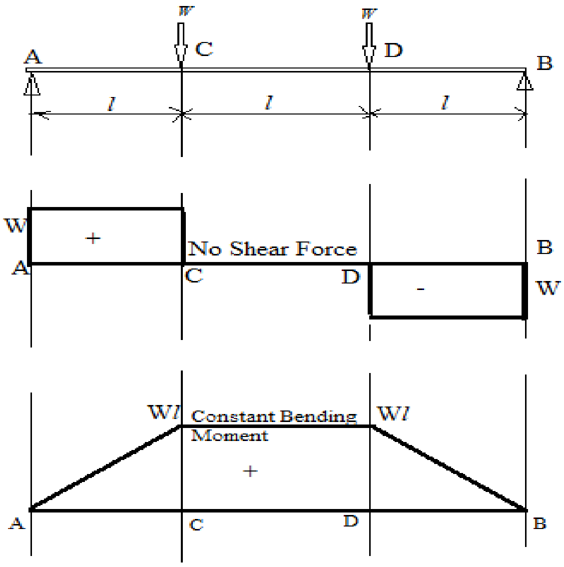
Due to these external loads, bending moments and shear forces are set-up at any point along the length of beam. Hence the beam has to resist the action of bending moment and shear force.

- The longitudinal stress produced at any section to resist the bending is known as the **bending stress or flexure.**



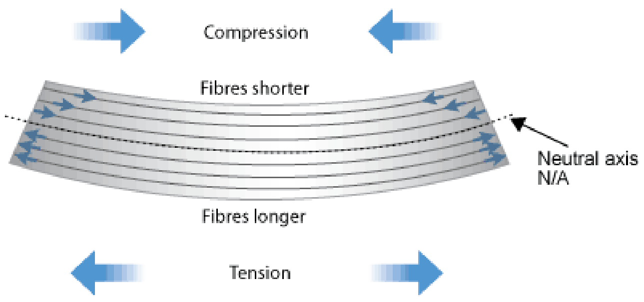
Pure bending or simple bending: If a length of beam is subjected to constant bending moment and no shear force, then stresses will be set-up in that length of the beam due to Bending Moment only and that length of beam is said to be pure bending or simple bending. Stresses are set-up in that length of the beam is known as Bending Stresses.

When a beam is bent due to application of constant bending moment, without being subjected to shear, it is said to be in a state of **simple bending or pure bending**.

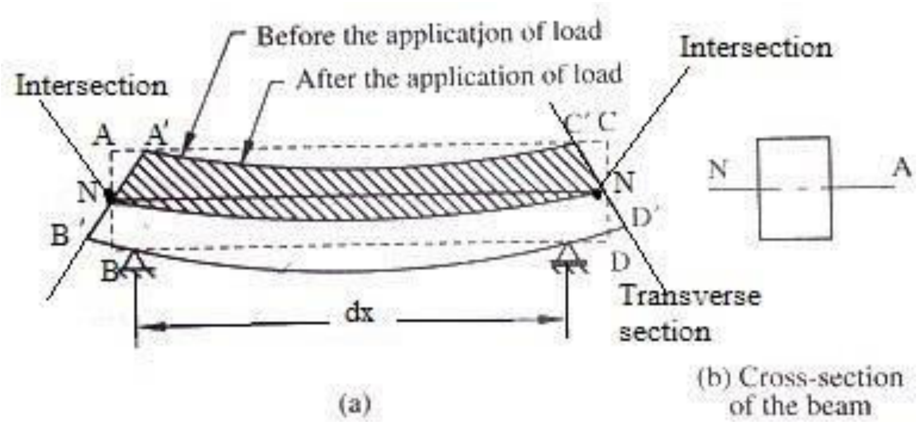

W
W
A
C
D
B
l
l
l
W
+
A
No Shear Force
C
D
B
-
W
Wl
Constant Bending
Moment
Wl
+
A
C
D
B

Theory of Simple Bending

- Consider a small length of simply supported beam subjected to bending moment.
- Consider two sections AB and CD, which are normal to axis of N-N.
- Due to action of bending moment, small length dx will deform.
- Layer of beam, which were originally of same length do not remain of same length.
- Top layer (layer above N-N) deformed from AC to A'C'. This top layer of beam has suffered compression and reduced to A'C'.



- Bottom layer (layer below N-N) deformed from BD to B'D'. This Bottom layer of beam has suffered Tension and elongated to B'D'.
- Between top and bottom of the beam, there will be longer which is neither shortened nor elongated. This layer is known as neutral layer or neutral surface.
- The line of intersection of neutral layer on a cross section (or with transverse section) of the beam is known as neutral axis (N.A.)



- Top layer has been shortened maximum, that means compressive will be maximum at the top layer.

- The amount by which layer is compressed or stretched, depending upon the position of layer with respect to N-N.
- This theory is called theory of simple bending.

Assumptions in the theory of simple of bending

1. Material of beam is perfectly homogeneous (same material throughout) and isotropic (equal elastic property in all direction).
2. The beam material is stressed within its elastic limit and thus obeys Hooke's law.
3. The value of young's modulus (E) is same in tension and compression.
4. The transverse sections, which were plane before bending, remain plane after bending also.

   (i.e. Entire beam cross-section is assumed to rotate about neutral axis.

5. Each layer of beam is free to expand or contract independently of the layer, above or below it.
6. The radius of curvature of the beam is very large in comparison to cross-sectional dimension of the beam.
7. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.

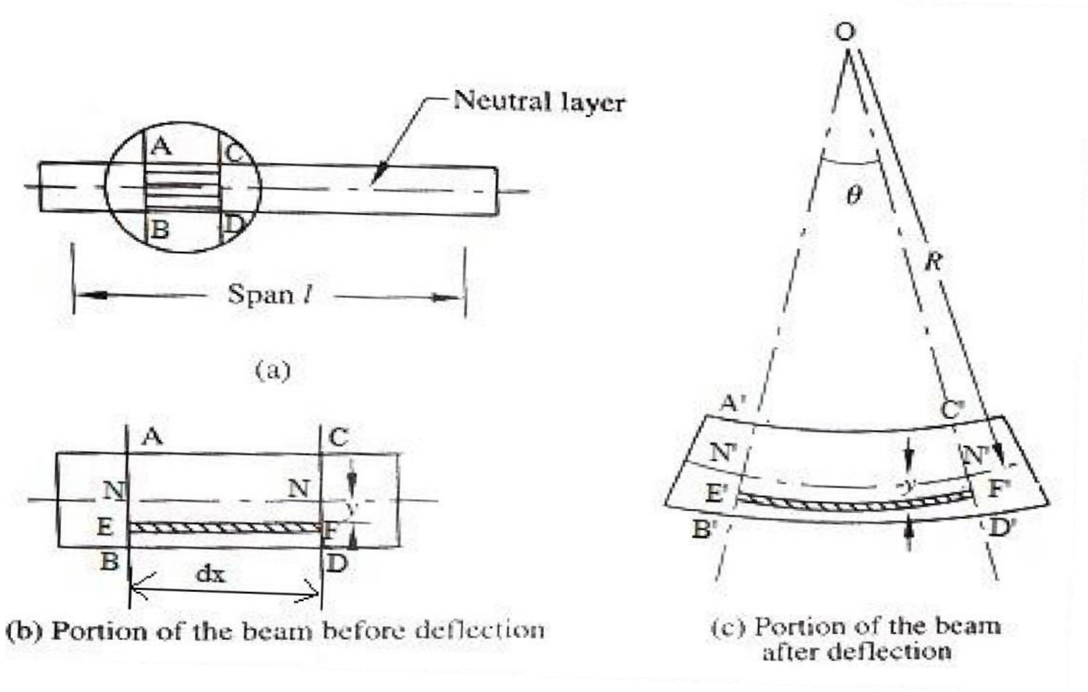
Derivation of bending equation and bending stress

Consider a portion of uniform beam subjected to pure bending. Due to action of bending, part of length dx will be deformed.

Let,

R = Radius of curvature of beam or Radius of neutral layer N'N' θ =
Angle subtended at o by A'B' and C'D'

Now consider a layer EF at a distance of y below the neutral layer NN, after the bending this layer will be elongated to E'F'



(b) Portion of the beam before deflection

(c) Portion of the beam after deflection

E'F' = NN = dx where EF = original of layer NN = length of neutral layer

After bending, the length of neutral layer N'N' will remain unchanged. But length of layer E'F' will increase.

NN = N'N' = dx

For ON'N' Angle = ______ θ = ____

N'N' = R. θ

dx = R. θ
EF = R.
θ

For OE'F' Angle = ______ θ = ______

E'F' = (R + y). θ

Increase in length of the layer EF = E'F' – EF = (R + y). θ – R. θ = y. θ

Strain in layer EF = ______________ = ε =

As R is constant, hence strain in a layer is proportional to its distance from neutral axis.

Young's modulus =

Stress = Young's modulus × Strain

$\sigma = E \times \varepsilon$

$\sigma = E \times$

σ = y × (_)   σ y

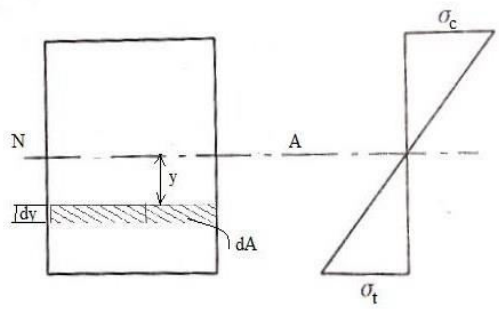
Since E & R both are constant, therefore stress in any layer is directly proportional to the distance of layer from the neutral layer.

— —

- Maximum compressive stress will be experienced by top most fibre.
- Maximum tensile stress will be experienced by bottom most fibre.

Neutral Axis The neutral axis of any transverse section of beam is defined as the line of intersection of the neutral layer with transverse.

We have seen,



- If a section of beam is subjected to pure sagging moment, then the stress will be compressive at any point above the neutral axis
- And tensile below the neutral axis
- There will be no stress at neutral axis
- Stress at a distance y of neutral axis   σ = y × (

  —)

Let N.A. is neutral axis

Consider a small layer at a distance of y from N.A. Let
= Area of that layer

Force on this layer = Stress on layer Area of layer

= (y ×_)

Total force on beam =   —

= —

But for pure bending, there is no force on the section of beam

—   = 0

E & R both are constant and it can’t be 0

= 0

- It represents moment of area dA about N.A.
- We know that moment of any area about an axis passing through its centroid, is also equal to

zero.

- Hence neutral axis coincides with centroidal axis.
- Centroidal axis of a section gives the position of N.A.

Moment of Resistance

- Due to pure bending, the layers above the N.A. are subjected to compressive stresses whereas layer below the N.A. are subjected to tensile stresses.
- These stresses form a couple (forces will act on these layers) whose moment must be equal to external moment (M).

The moment of this couple, which resist the external bending moment, is known as moment of resistance.

Force on layer at a distance y from N.A. = (y ×) —

Moment of this force about N.A. = Force on layer y

= (y ×) y

= —

Moment of forces on the section of beam (or moment of resistance) = —

Let M is external moment applied on this section.

For equilibrium the moment of resistance offered by section should be equal to external bending moment.

M = —

= Second moment of area or moment of inertia of section about N.A. M =

— — —

We know that — —

— — —

This equation is known as bending equation or Bernoulli-Euler equation.

Section modulus

It is the ratio of moment of inertia of a section about the neutral axis to the distance of outermost layer from the neutral axis. It is denoted by Z.

Z =

I = Moment of inertia about N.A.

Distance of outermost layer from the neutral axis

Maximum bending moment (or moment of resistance offered by the section)

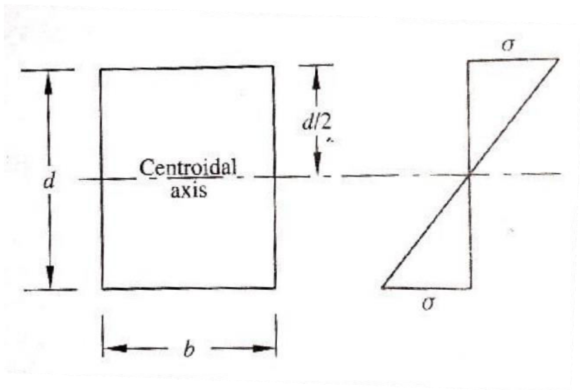
M = Z

- Moment of resistance offered by the section is maximum when section modulus Z is maximum.
- Section modulus represents strength of the section.

Section modulus for various shapes

a) Rectangular Section:

Moment of inertia of a rectangular section about an axis through its C.G.(or through N.A.)



I =

Distance of outermost layer from N.A.

Z =

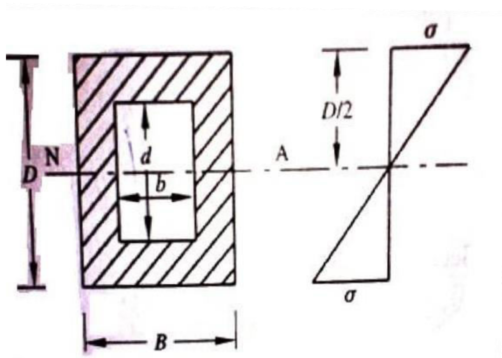
Z =

b) Hollow rectangular section:

~~I =~~

Z =

Z =

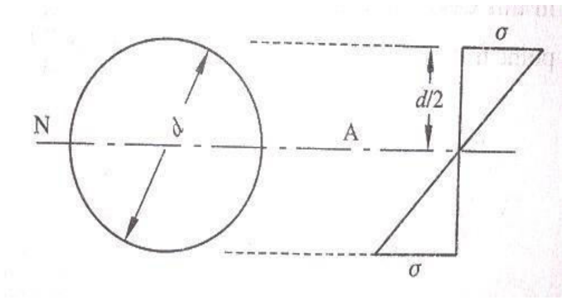


c) Solid Circular Section:

I =

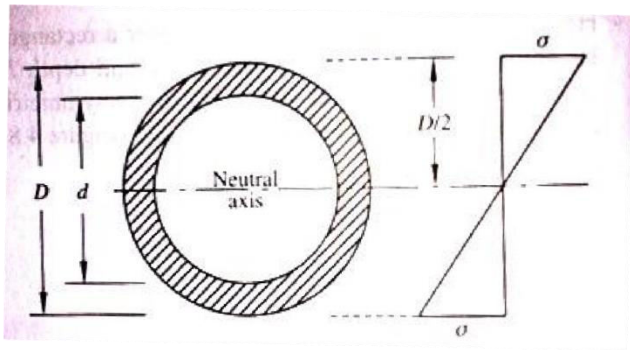
Z = =

Z =



d) Hollow Circular Section:

I = =



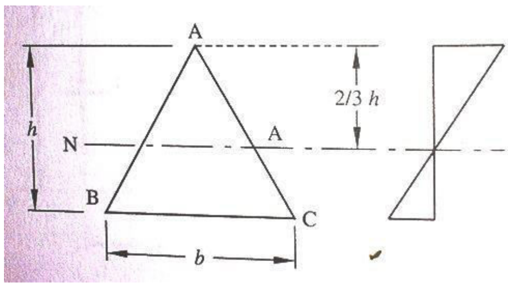
e) Triangular Section:

b = base width of triangle h
= height of triangle

I =

Z = =

Z =



Strength of section: It means the moment of resistance offered by the section. M = σ .
Z

- Moment of resistance depends upon section modulus.
- Greater the value of section modulus, stronger will be the section.

σ =

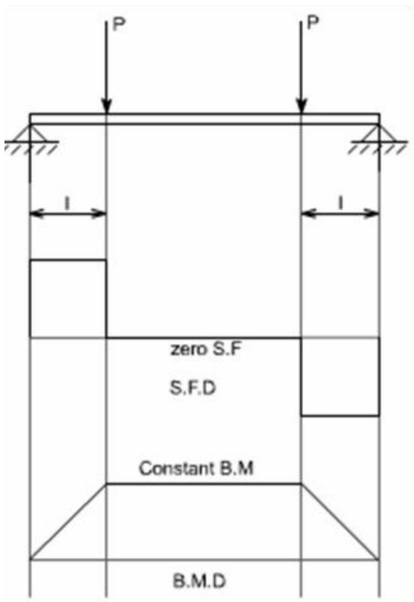
σ = σ =

- If section modulus is small, then stress will be more.

PART A: FLEXURAL STRESSES (CONCEPTS REVIEW)

The normal stresses developed in structural members due to bending are called flexure stresses. When a beam member is subjected to transverse loads, it bends. Depending on the loading, twisting and buckling effects may also occur. In this section, we are interested to study the bending effects alone, and not the combined effects of bending twisting and buckling.

Theory of simple bending or pure bending



Due to the eternal loading, the internal reactions developed on any cross-section of a beam may consist of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the only bending effects are prominent, we assume that the loading is such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the beam member.

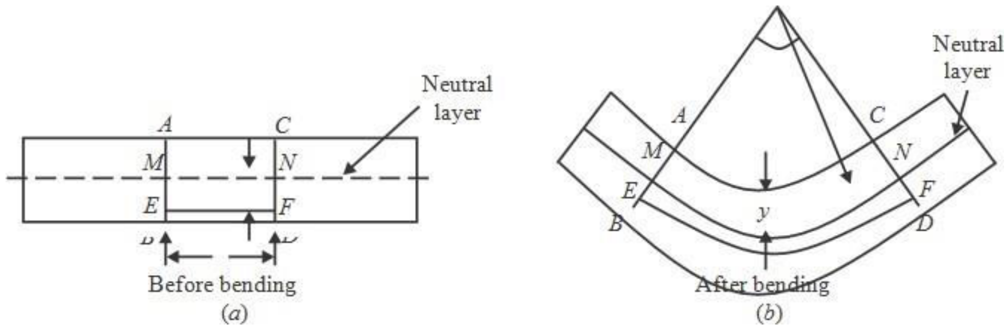
When the loading is such that a beam section experiences only constant moment and has zero shear force, then it is said to be subjected to **pure bending** or **simple bending**. For example, for the loading pattern shown, the span between the two point loads is in the state of pure bending.

Assumptions for the theory of pure bending:

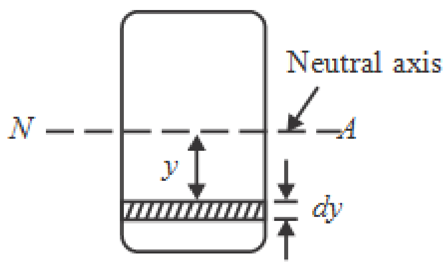
1. The beam is initially straight and has a constant cross-section.
2. Beam material is homogeneous and isotropic
3. The beam has a longitudinal plane of symmetry.
4. Resultant of the applied loads lies in the plane of symmetry.
5. The geometry of the overall member is such that bending not buckling is the primary cause of failure.
6. Elastic limit is nowhere exceeded and ‘E' is same in tension and compression.
7. Plane cross - sections remains plane before and after bending.
8. Shear force at the beam cross-section is zero.

Derivation of the bending equation

Consider the beam section as shown in the figure below. Consider any two normal sections *AB* and *CD* of a beam at small distance δL apart (that is, $AC = BD = \delta L$). Let *AB* and *CD* intersect neutral layer at the points *M* and *N* respectively. When the beam is subjected to bending as shown, the top layers of the beam are subjected to compression and the bottom layers are subjected to tension.
The neutral axis or the neutral layer is the layer which is neither subjected to tension or compression.



$\sigma/y = \boldsymbol{E/R}$



Figure

Let;

M = bending moment acting on beam

θ = Angle subtended at centre by the arc.

R = Radius of curvature of neutral layer *M' N'* .

At any distance '*y*' from neutral layer *MN*, consider layer *EF*.

As shown in the figure the beam because of sagging bending moment. After bending, *A' B'*, *C' D'* , *M' N'* and *E'F'* represent final positions of *AB*, *CD*, *MN* and *EF* in that order. When produced, *A' B'* and *C' D'* intersect each other at the *O* subtending an angle θ radian at point *O*, which is centre of curvature. As *L* is quite small, arcs *A' C'* , *M' N'* , *E' F'* and *B' D'* can be taken as circular.

Now, strain in layer *EF* because of bending can be given by *e* = (*E' F'* - *EF*)/*EF* = (*E' F'* - *MN*)/*MN*

As *MN* is the neutral layer, *MN* = *M' N'*

$$e = \frac{E'F' - M'N'}{M'N'} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y\theta}{R\theta} = \frac{y}{R}$$

…………………………………………(i)

Let; σ = stress set up in layer EF because of bending

E = Young's modulus of material of beam.

$$E = \frac{\sigma}{e} \qquad \text{or,} \quad e = \frac{\sigma}{e}$$

……………………………………………(ii)

Equate the equation (i) and (ii);

$$\frac{y}{R} = \frac{\sigma}{E}$$

……………………………………………(iii)

At distance '*y*', let us consider an elementary strip of quite small thickness *dy*. We have already assumed that 'σ ' is bending stress in this strip.

Let *dA* = area of the elementary strip. Then, force developed in this strip = σ.*dA*.

Then the, elementary moment of resistance because of this elementary force can be given by *dM* = *f.dA.y*

Total moment of resistance because of all such elementary forces can be given by

$$\int dM = \int \sigma \times dA \times y$$
$$M = \int \sigma \times dA \times y$$

From the Equation (iii),

$$\sigma = y \times \frac{E}{R}.$$

By putting this value of f in Equation (iv), we get

$$M = \int y \times \frac{E}{R} \times dA \times y = \frac{E}{R} \int dA \times y^2$$

But

$$\int dA \cdot y^2 = 1$$

where I = Moment of inertia of whole area about neutral axis N-A.

$$M = (E/R)\ .\ I$$
$$M/I = E/R$$
$$M/I = \sigma/y = E/R$$

Where,

M = Bending moment

I = Moment of Inertia about axis of bending that is; I_{xx}

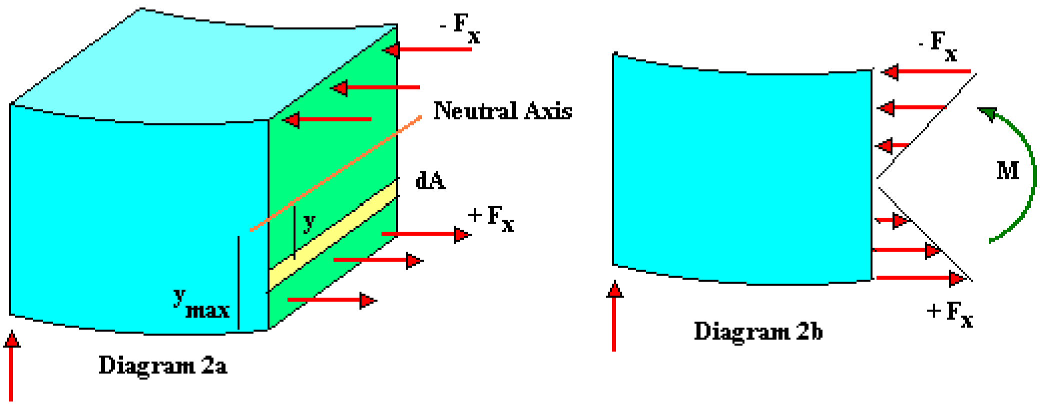
y = Distance of the layer at which the bending stress is consider

(We take always the maximum value of y, that is, distance of extreme fiber from N.A.)

E = Modulus of elasticity of beam material.

R = Radius of curvature

Determination of bending stresses



The above figure shows the distribution of bending stresses in the beam subjected to moment. The bending stress distribution along the cross-section is given in the second figure.

The bending stress, □, is calculated by using the bending equation as

We can see that the bending stress is directly proportional to ‘y’, the distance of the fibre in consideration from the neutral axis. Thus, the bending stress is maximum at the extreme fibres of the beams.

Section modulus

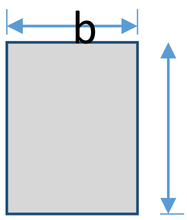
The section modulus is defined as:

where I is the moment of inertia taken about the neutral axis. Thus, in the bending equation, the maximum bending stress or,

Section modulus is a pure geometric property for a given cross-section. It is most useful in the design of beams or flexural members. For general design, the elastic section modulus is used, applying up to the yield point for most metals and other common materials. It is also often used to determine the yield moment (M_y) such that $M_y = Z\,\sigma_y$, where σ_y is the yield strength of the material. The moment of resistance offered by the section is maximum if the section modulus is maximum.
Thus, ***the section modulus thus represents the strength of the section***.

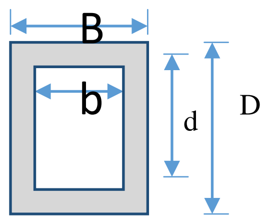
Section modulus for some of the typical sections is as shown below:

A) Rectangular section



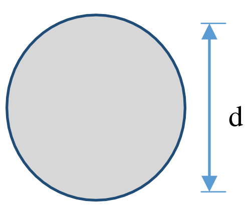
$$I = \frac{bd^3}{12};\ Z = \frac{bd^2}{6}$$

B) Hollow Rectangular Section



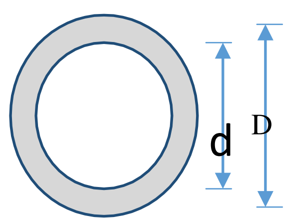
$$I = \frac{1}{12}\left[BD^3 - bd^3\right];\ Z = \frac{1}{6D}\left(BD^3 - bd^3\right)$$

C) Circular Section



$$I = \frac{\pi d^4}{64};\ Z = \frac{\pi d^3}{32}$$

D) Hollow Circular Section

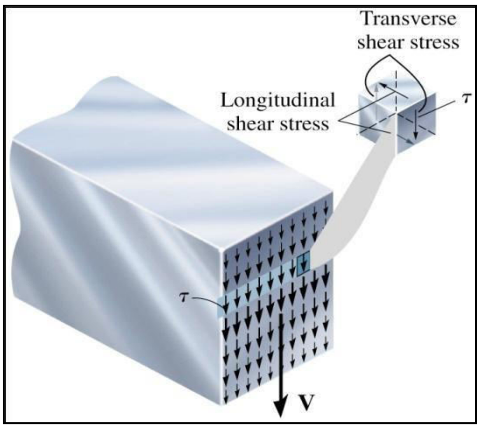


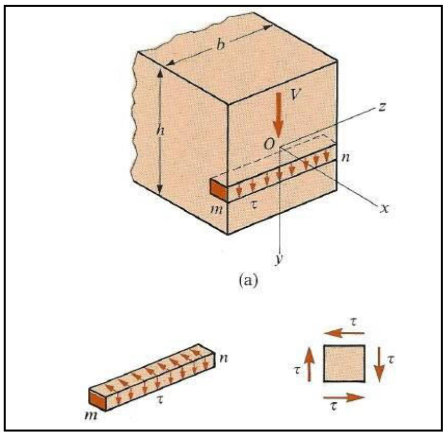
$$I = \frac{\pi}{64}\left[D^4 - d^4\right];\ Z = \frac{\pi}{32D}\left[D^4 - d^4\right]$$

PART B: SHEAR STRESSES

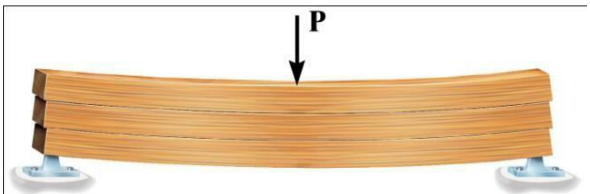
Transverse and Longitudinal Shear Stresses in Beam Sections

In the previous section, we computed the bending stress (normal stress) on a cross section of a beam when the beam members are subjected to simple or pure bending. This means that we assumed bending moment to be constant and the shear force on that section to be zero. However, when beam members are subjected to loads, in general we find that bending moment and shear forces exist together at a section. In those sections, where the bending is associated with shear force, we need to compute the shearing stress as well. In this section, we shall learn how to compute the value of the shearing stress and their distributions over a cross-section.



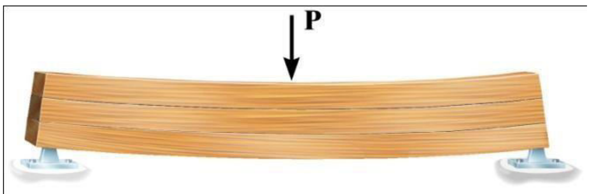


As we have studies from the earlier chapter of stresses, the transverse shear stresses at a section will always be accompanied by a complimentary horizontal shear stresses acting on the longitudinal layers of the beam. This is depicted in the above figures. This can also be demonstrated by applying transverse load to a set-up of layers of wooden boards, not bonded together and bonded together, as shown below.



Boards not bonded together

(a)



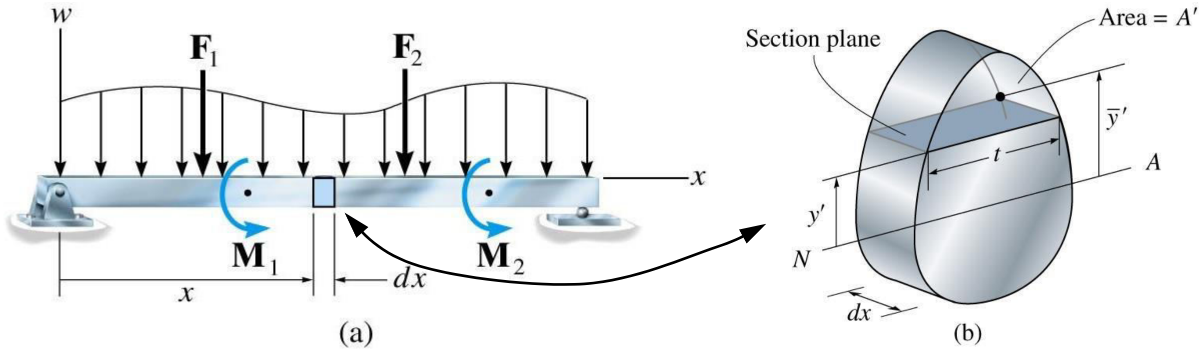
Boards bonded together

(b)

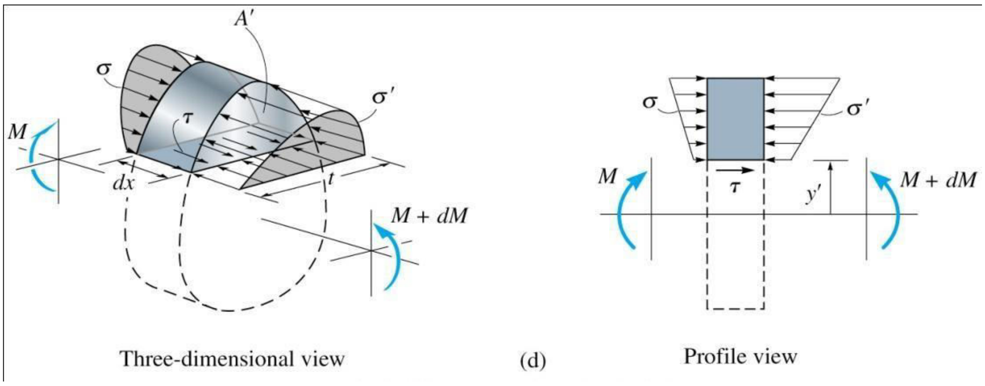
When boards are not glued together, they slip relative to each other at the layers of their separation. Thus, bending action tends to produce longitudinal displacement in the material. When boards are glued together, the slippage action is prevented and shear stress is developed at the inter layer surfaces.

Shear Stress at a Section in a Beam

Consider a section of a beam subject to a general case of loading as shown in the figure below.



The normal stresses due to bending (flexure) on the sections are as shown below.



By applying the force equilibrium in the longitudinal direction, we get

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau (t\,dx) = 0$$

$$\int_{A'} \left(\frac{M + dM}{I} \right) y\,dA - \int_{A'} \left(\frac{M}{I} \right) y\,dA - \tau (t\,dx) = 0$$

Simplifying and substituting, dM/dx = V, the shear force, we get

$$\tau = \frac{1}{It} \left(\frac{dM}{dx} \right) \int_{A'} y\,dA$$

$$\tau = \frac{VA'\bar{y}}{It}$$

The above shear-formula has to be understood as follows. The above formula gives the shear stress at the layer of the cross-section of the beam, located at a distance of y from the neutral axis.

V is the shear force acting at that particular section of the beam. I

is the moment of inertia of the entire cross-section of the beam.

‘t’ is the width of the cross-section, at the layer of interest, i.e. at a distance of y from the neutral axis.

A□ is the portion of the area of the cross-section of the beam, above the layer of interest.

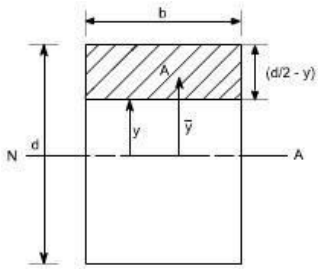
$\bar{y}$ is the distance of the centroid of the area A' from the neutral axis.

Shearing stress distribution in typical cross-sections

Let us consider few examples to determine the sheer stress distribution in a given X- sections

Shear stress distribution in beams of a rectangular section

Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$\text{for this case, } A = b\left(\frac{d}{2} - y\right)$$

$$\text{While} \quad \bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$$

$$\text{i.e} \quad \bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right) \text{ and } z = b\,;\, I = \frac{b.d^3}{12}$$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y. The maximum value of shear stress would obviously beat the location y = 0.

$$\text{Such that} \quad \tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$$
$$= \frac{3.F}{2.b.d}$$
$$\text{So} \quad \boxed{\tau_{max} = \frac{3.F}{2.b.d}} \text{ The value of } \tau_{max} \text{ occurs at the neutral axis}$$

The mean shear stress in the beam is defined as

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$
$$\text{So} \quad \tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$$

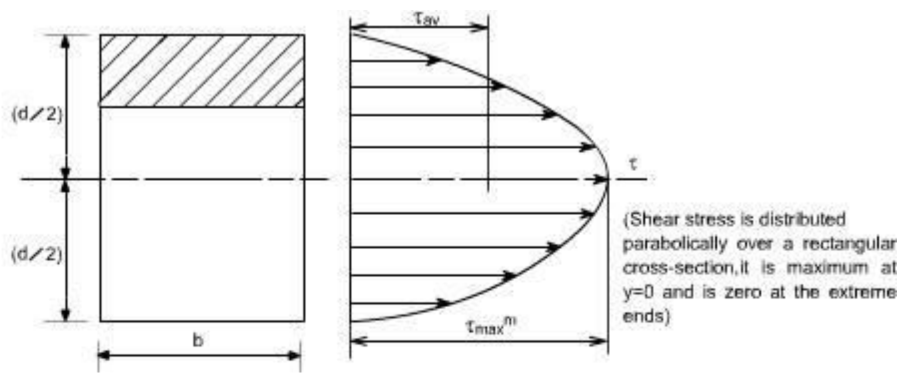
This shows that there is a parabolic distribution of shear stress with y. The maximum value of shear stress would obviously beat the location y = 0

$$\text{Such that} \quad \tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$$
$$= \frac{3.F}{2.b.d}$$
$$\text{So} \quad \boxed{\tau_{max} = \frac{3.F}{2.b.d}} \text{ The value of } \tau_{max} \text{ occurs at the neutral axis}$$

The mean shear stress in the beam is defined as

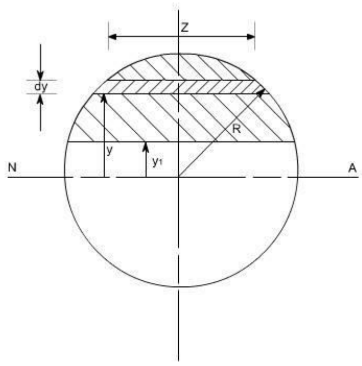
$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$
$$\text{So} \quad \tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$$

Therefore the shear stress distribution is shown as below.



It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

Shear stress distribution in beams of circular cross-section



Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.

Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\,\bar{y}}{ZI} = \frac{FA \int y\, dA}{ZI}$$

Where $\int$ *y dA* is the area moment of the shaded portion or the first moment of area.

Here in this case 'dA' is to be found out using the Pythagoras theorem

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$I_{N.A}$ for a circular cross-section $= \dfrac{\pi R^4}{4}$

Hence,

$$\tau = \frac{FA\bar{y}}{ZI} = \frac{F}{\dfrac{\pi R^4}{4} 2\sqrt{R^2 - y^2}} \int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4\,F}{\pi R^4 \sqrt{R^2 - y^2}} \int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4\,F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{max^m}|y_1 = 0 = \frac{4\,F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$

so this again a parabolic distribution; maximum at the neutral axis

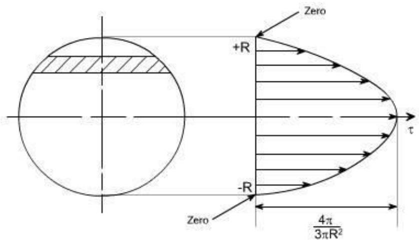
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

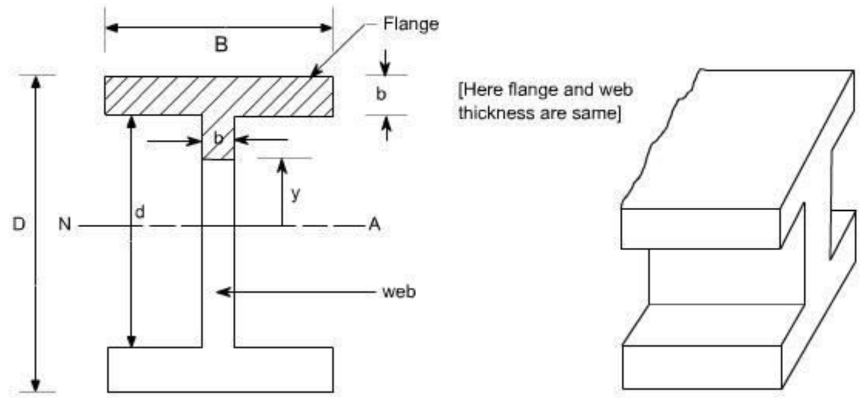
$$\boxed{\tau_{max^m} = \frac{4}{3}\tau_{avg}}$$

The distribution of shear stresses is shown below, which indicates a parabolic distribution



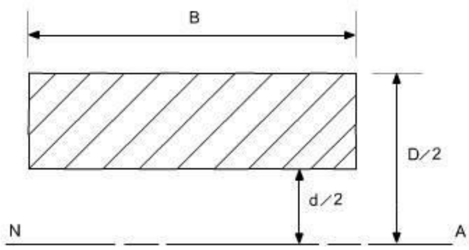
Shear stress distribution in beams of I - section

Consider an I - section of the dimension shown below



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Flange:



$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

Distance of the centroid of the flange from the N.A

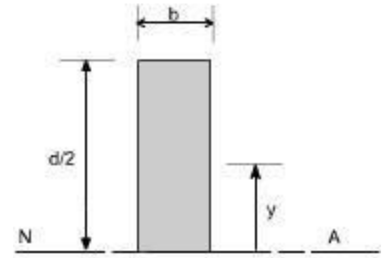
$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\Big|_{\text{Flange}} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$

Web area:



Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of t substitute in the above relation.

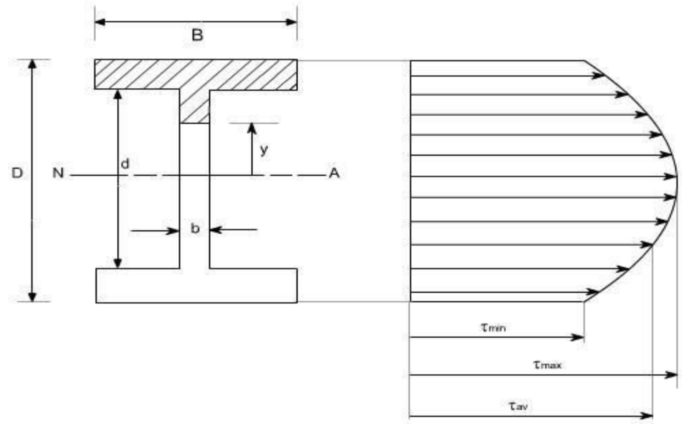
y = 0 at N. A. And y = d/2 at the tip. The maximum shear stress is at the neutral axis. i.e. for the condition y = 0 at N. A.

$$\tau_{max} \text{ at } y = 0 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

$$\tau_{min} \text{ at } y = d/2 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right)\right]$$

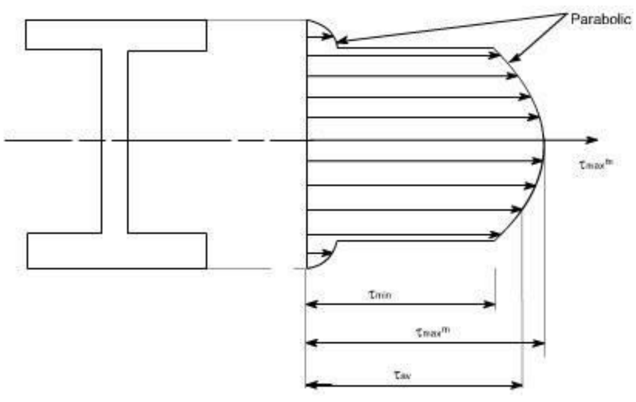
The distribution of shear stress may be drawn as below, which clearly indicates a parabolic distribution.



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.



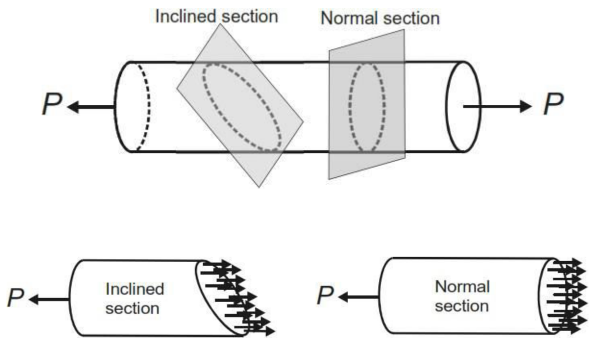
This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

UNIT - IV

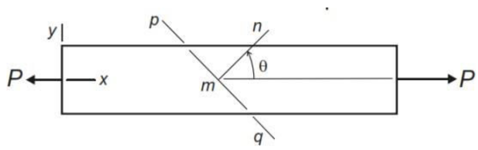
Part A: Principal Stresses and Strains

Normal and shear stresses on inclined sections

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an "inclined" (as opposed to a "normal") section through the bar.



Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed. Specify the orientation of the inclined section pq by the angle θ between the x axis and the normal to the plane.



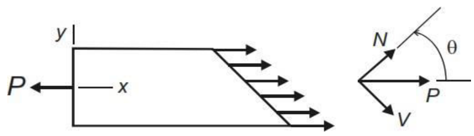
"Normal section" θ = 0°,

Top face θ = 90° ,

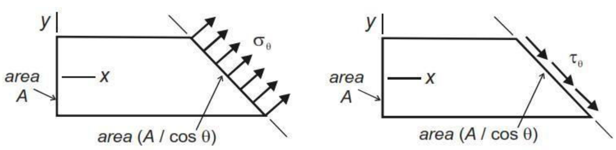
Left face = 180° ,

Bottom face θ = 270° or -90°

The force P can be resolved into components: (A) Normal force N perpendicular to the inclined plane, N = P cos θ (B) Shear force V tangential to the inclined plane V = P sin θ



If we know the areas on which the forces act, we can calculate the associated stresses.



$$\sigma_\theta = \frac{Force}{Area} = \frac{N}{Area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta) \quad ☆$$

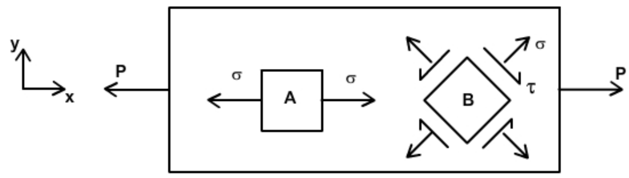
$$\tau_\theta = \frac{Force}{Area} = \frac{-V}{Area} = \frac{-P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta) \quad ☆$$

State of Stress (Compound Stress)

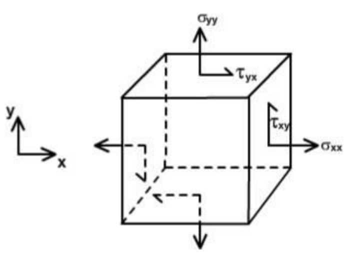
Depending upon the state of stress at a point, we can classify it as uniaxial (1D), biaxial (2D) and triaxial (3D) stress.

Uniaxial Stress



Consider a bar under a tensile load P acting along its axis as shown in the figure. Take an element A which has its sides parallel to the surfaces of the bar. It is clear that the element has only normal stress along only one direction, i.e., x axis and all other stresses are zero. Hence it is said to be under uni-axial stress state. Now consider another element B in the same bar, which has its slides inclined to the surfaces of the bar. Though the element has normal and shear stresses on each face, it can be transformed into a uni-axial stress state like element A by transformation of stresses (will be discussed in later section). Hence, if the stress components at a point can be transformed into a single normal stress (principal stress as will be discussed later), then, the element is under uni-axial stress state.

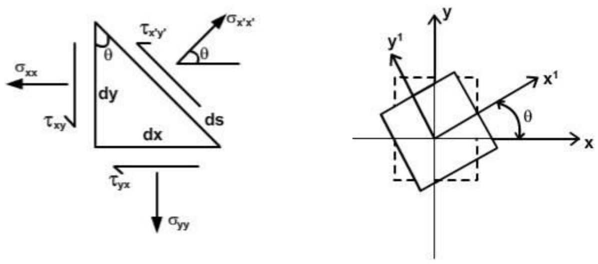
Two dimensional stress (Plane stress)



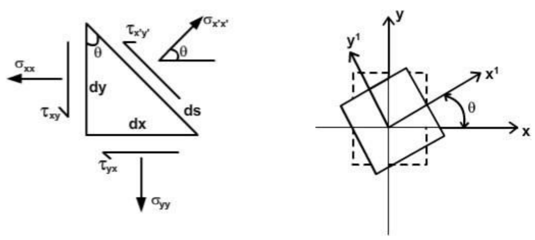
When the cubic element is free from any of the stresses on its two parallel surfaces and the stress components in the element cannot be reduced to a uni-axial stress by transformation, then, the element is said to be in two dimensional stress/plane stress state. Thin plates under mid plane loads and the free surface of structural elements may experience plane stresses as shown below.

Transformation of plane stress

Though the state of stress at a point in a stressed body remains the same, the normal and shear stress components vary as the orientation of plane through that point changes. Under complex loading, a structural member may experience larger stresses on inclined planes then on the cross section. The knowledge of maximum normal and shear stresses and their plane's orientation assumes significance from failure point of view. Hence, it is important to know how to transform the stress components from one set of coordinate axes to another set of axes that will contain the stresses of interest.



Consider a prismatic element with sides dx, dy and ds with their faces perpendicular to y, x and x' axes, respectively, as shown. Take thickness of the element as t.



$\sigma_{x'x'}$ and $\tau_{x'y'}$ are the normal and shear stresses acting on a plane inclined at an angle θ, measured counter-clockwise from the x–plane. Applying equilibrium conditions and solving we get

$$\sigma_{x'x'} = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy} \cos 2\theta$$

Replacing θ by $90 + \theta$ we get the normal stress along y' direction as

$$y \square y \square \quad \square \frac{\square}{2} \square \quad \square \frac{\square}{2} \square \quad \square \quad \frac{\sin 2\square}{xy}$$

The above equations are the transformation equations for plane stress using which the stress components on any plane passing through the point can be determined. Notice here that,

$$\sigma_{x'x'} + \sigma_{y'y'} = \sigma_{xx} + \sigma_{yy}$$

Invariably, the sum of the normal stresses on any two mutually perpendicular planes at a point has the same value. This sum is a function of the stress at that point and not on the orientation of axes. Hence, this quantity is called *stress invariant* at that a point.

Principal stresses and maximum shear stress

From transformation equations, it is clear that the normal and shear stresses vary continuously with the orientation of planes through the point. Among those varying stresses, finding the maximum and minimum values and the corresponding planes are important from the design considerations.

By differentiating the expression for $\sigma_{x'x'}$ with respect to θ and equating to zero, we get

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

For this value of θ, referred to as θ_p, the shear stress $\tau_{x'y'} = 0$. Such a plane where normal stress is maximum/minimum and the shear stress is zero, is called a principal plane. And the corresponding normal stress is called as the principal stress.

Here, θ_p, has two values θ_{p1} and θ_{p2} that differ by 90^o, with one value between 0^o and 90^o and the other between 90^o and 180^o. These two values define the principal planes that contain maximum and minimum stresses. Substituting these two θ values in equation for $\sigma_{x'x'}$, the maximum and minimum stresses, also called as principal stresses, are obtained.

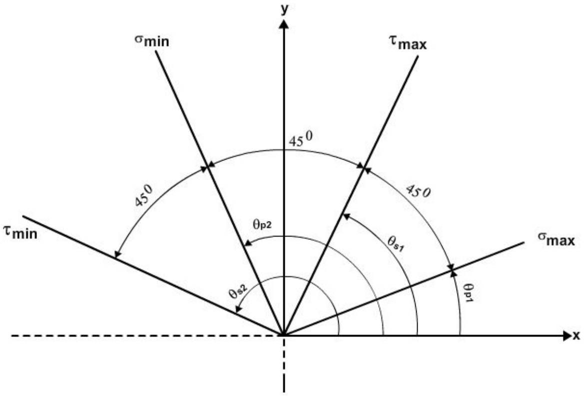
$$\sigma_{max}/\sigma_{min} = \sigma_{p1}/\sigma_{p2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

The plus and minus signs in the second term of the above equation, indicate the algebraically larger and smaller principal stresses, i.e. maximum and minimum principal stresses.

To get the maximum value of the shear stress, the derivative of $\tau_{x'y'}$ with respect to θ is equated to zero. We get,

$$\tan 2\theta_s = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}\right)$$

θ_s has two values, θ_{s1} and θ_{s2} that differ by 90^o, with one value between 0^o and 90^o and the other between 90^o and 180^o. Hence, the maximum shear stresses that occur on those two mutually perpendicular planes are equal in algebraic value and are different only in sign due to its complementary property. Also we can observe that $\tan 2\theta_p \tan 2\theta_s = -1$. Thus, the angles $2\theta_p$ and $2\theta_s$ differ by 90^o, and furthermore the angles θ_p and θ_s differ by 45^o.



A typical example of the inclinations of the principal planes and the planes of maximum and minimum shear stress are shown in the above figure.

The principal planes do not contain any shear stress on them, but the maximum shear stress planes may or may not contain normal stresses as the case may be. Maximum shear stress value is found out by substituting θs values in equation for $\tau_{x'y'}$. And thus we obtain,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2}$$

We can also show that

$$\tau_{max/min} = \pm \frac{\sigma_{max} - \sigma_{min}}{2} = \pm \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

Mohr's circle for plane stress

The transformation equations of plane stress, i.e. expressions for $\sigma_{x'x'}$ and $\tau_{x'y'}$ can be represented in a graphical form which is popularly known as Mohr's circle. Though the transformation equations are sufficient to get the normal and shear stresses on any plane at a point, with Mohr's circle one can easily visualize their variation with respect to plane orientation θ. Besides stress plots, Mohr's circles are used to plot strains, moment of inertia, etc., which follow the same transformation laws
as do stresses.

Let us consider the transformation equations,

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

By algebraic manipulation and eliminating θ, from the above equations we get

$$\left(\sigma_{x'x'} - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2$$

This is the equation of the Mohr’s circle. For simple representation of above equation, the following notations are used.

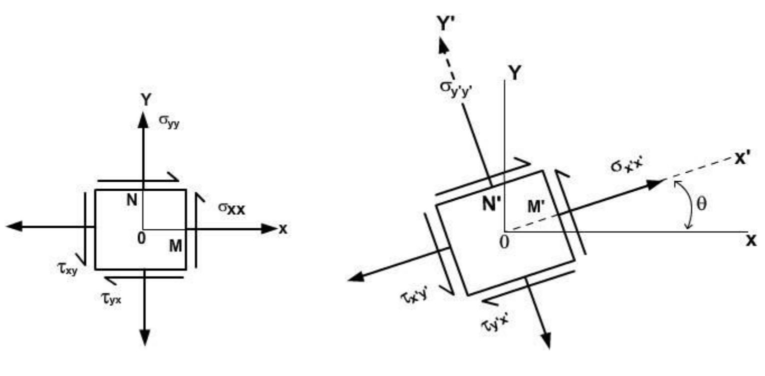
$$\sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2}; \quad R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Thus, the Mohr’s circle can be simplified as

$$\left(\sigma_{x'x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

The equation represents a circle, when we plot $\sigma_{x'x'}$ on the x-axis (horizontal axis) and $\tau_{x'y'}$ on the y- axis (vertical axis). The centre of this circle is C $(\sigma_{avg}, 0)$ and radius is R.

Construction of the Mohr’s circle (Plane Stress)

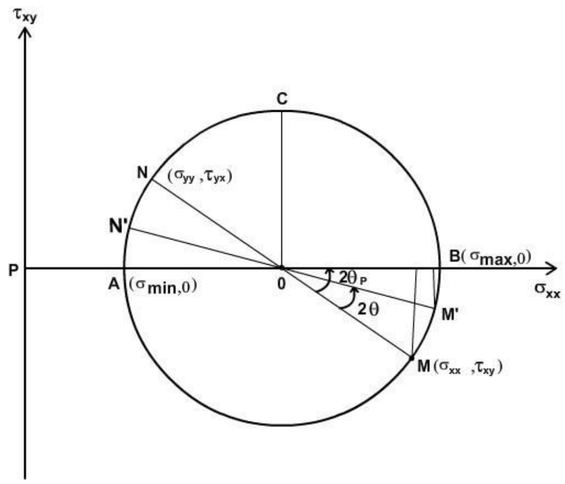


Sign convention: Tension is positive and compression is negative. Shear stresses causing clockwise moment about O are positive and counter-clockwise negative. Hence, τ_{xy} is negative and τ_{yx} is positive.

Mohr's circle is drawn with the stress coordinates $\sigma_{x'x'}$ as its abscissa and $\tau_{x'y'}$ as its ordinate, and this plane is called the stress plane. The plane on the element in the material with xy coordinates is called the physical plane. Stresses on the physical plane M is represented by the point

M on the
stress plane with σ_{xx} and τ_{xy} as its coordinates.

Stresses on the physical plane N, which is normal to M, is represented by the point N on the stress plane with σ_{yy} and τ_{yx}. The intersecting point of line MN with abscissa is taken as O, which turns out to be the centre of circle with radius OM.



Now, the stresses on a plane which makes θ inclination with x axis in physical plane can be determined as follows. Let that plane be M'. An important point to be noted here is that a plane which has a θ in physical plane will make 2θ inclination in stress plane.

Hence, rotate the line OM in stress plane by 2θ inclination counter-clockwise to obtain the plane M'.

Points A and B on Mohr's circle do not have any shear components and hence, they represent the principal stresses.

The principal plane orientations can be obtained in Mohr's circle by rotating the line OM by $2\theta_p$ and $2\theta_p+180^o$ clockwise or counter-clockwise as the case may be (here it is counter-clock wise) in order to make that line be aligned to the horizontal axis, $\tau_{xy} = 0$.

These principal planes on the physical plane are obtained by rotating the plane m, which is normal to x axis, by θ_p and θ_p+90^o in the same direction as was done in stress plane.

The maximum shear stress is defined by OC in Mohr's circle.

It is important to note the difference in the sign convention of shear stress between analytical and graphical methods, i.e., physical plane and stress plane. The shear stresses that will rotate the element in counter-clockwise direction are considered positive in the analytical method and negative in the graphical method. It is assumed this way in order to make the rotation of the elements consistent in both the methods.

Plane Strain

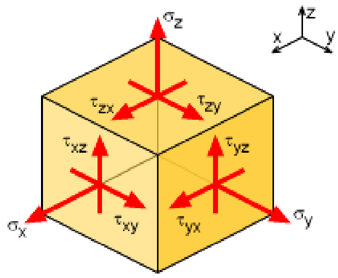
For the most generalized state of stress, the relation with strains is given by Hooke's law as:

$$\varepsilon_{xx} = \frac{1}{E}\left[\,\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz}\,\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\,\sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xx}\,\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\,\sigma_{zz} \quad \sigma_{xx} \quad \sigma_{yy}\,\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \quad \gamma_{yz} = \frac{\tau_{yz}}{G}; \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$



Plane Strain

The normal and shear strains at a point in body vary with direction. For plane strain, the strain components are

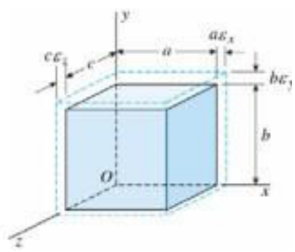
$\varepsilon_x, \quad \varepsilon_y, \quad \gamma_{xy}$ and $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

plane stress : $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

but $\varepsilon_z \neq 0$

plane strain : $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

but $\sigma_z \neq 0$

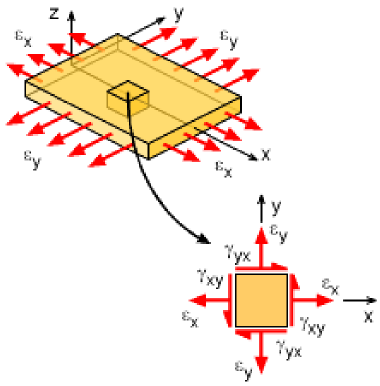


Plane stress and plane strain do not occur simultaneously, in general.

Only in special cases: plane stress ==> plane strain

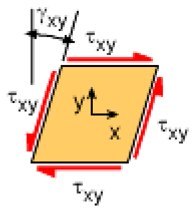
1. $\sigma_x = -\sigma_y \qquad \varepsilon_z = -\nu(\sigma_x + \sigma_y)/E = 0$
2. $\nu = 0 \qquad \varepsilon_z = 0$ for $\sigma_z = 0$

Plane Strain Element



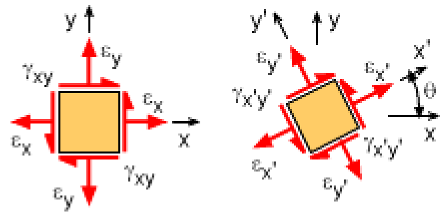
Similar to stress analysis at a point, strain can also be rotated to different directions and thus maximum shear and normal strains can be found. Strain at a given point on a two dimensional object can be seen as a small strain element with two normal strains and one shear strain, as shown in the diagram.

The sign convention for strains at a point is similar to stresses at a point. Normal tension strain in both the x and y direction are assumed positive. The positive shear strain direction is shown in the diagram at the left. Shear stresses act on four sides of the stress element, causing a pinching or shear action. Shear strains on all four sides are the same, thus $\gamma_{xy} = \gamma_{yx}$



Recall, the shear strain is actually defined as the angle of rotation or twist due to the shear stress. This angle is in radians and is shown in the figure.

Transformation of Plane Strain



Stains at a point (strain element) can be rotated to give a new strain state at any particular angle. The rotation angle, θ, is assumed positive using the right hand rule (counter-clockwise in the x-y plane is positive). The new coordinate system is labeled as x' and y'. The new rotated strains are shown in the diagram at the left. The shear strains, $\gamma_{x'y'}$ and $\gamma_{y'x'}$ are still equal.

As one may expect, the strain transformation equations are nearly identical to the stress transformation angles. The only difference is a factor of 2 for the shear strain. Other than that, the strain symbol, ε, can be simply substituted for σ to give the strain transformation equations (detail derivation is omitted). The final equations are,

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} - \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Principal Strains

Generally, the largest normal strain is of most interest. This can be found by taking a derivative of either the $\varepsilon_{x'x'}$ or $\varepsilon_{y'y'}$ strain with respect to θ and equating it to zero. This will give the principal rotation angle, θ_p, that will produce the principal (maximum and minimum) strains. The resulting equations for principal strains

$$\varepsilon_{p1}/\varepsilon_{p2} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

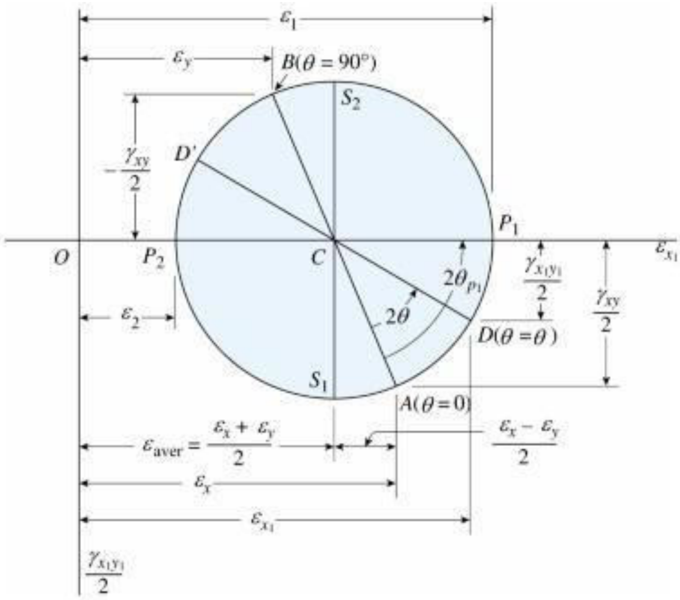
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}}$$

Maximum Shear Strain: Similar to the principal strains, the maximum shear strain can be determine by taking a derivative of $\gamma_{x'y'}$ strain with respect to θ and equating it to zero. This gives

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_s = -\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{\gamma_{xy}}\right)$$

Mohr's Circle for plane strain



Plane strains at a point can be measured by strain rosette, then the stresses at this point can be calculated, also the principal strains and principal stresses can be obtained.

Part B: Theories of Elastic Failure of Materials

Introduction

From the simple tension test on a bar of a material performed in a laboratory, one can determine the maximum load capacity of the material. Now, we will assume that in a complex loading condition also, the material has the same capability. This assumption forms the backbone of Failure theories. Concepts of Simple tension test and Principal stresses are the two main pre-requisites to understand the failure theories effectively.

Simple Tension Test

In Simple tension test material is pulled from both the ends, the elongation of material (strain) with respect to the load is noted. From such an observation one can easily determine maximum strength of the material. For ductile material *upper yield point* is considered to be maximum strength of material, while for brittle material it is taken as *ultimate strength* of the material. From the maximum strength value of the material, values of various other parameters can easily be calculated.Simple tension graph and *upper yield point* value for a ductile material case is shown in the figure below.

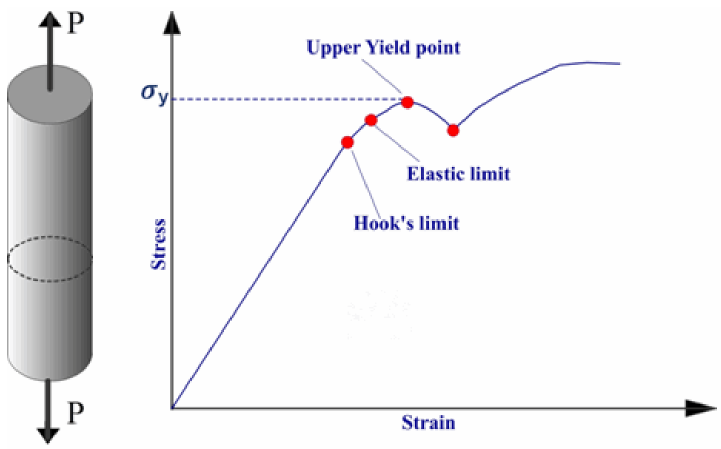


Fig. Simple tension test

Principal Stress

Principal stress is the maximum normal stress occurring at a given point. In order to find out this value easy way is to do a Mohr circle analysis. Once you know Principal stress values you can go ahead with failure theories. Figure below shows principal stress values induced at point in a 3 dimensional complex loading case.

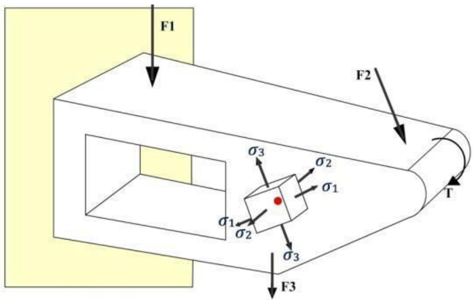


Fig. Principal stresses and planes

The Failure Theories

The interesting thing in the failure theories is that, just by looking at the name of the theory you will be able to formulate condition of failure in an actual case. Just make sure that your concept of STT and Principal stresses are clear. The theories along with its usability are given below.

Maximum principal stress theory [Rankine's Theory]

[Good for brittle materials*]

According to this theory when the maximum principal stress induced in a material under complex load condition exceeds the maximum normal strength in a simple tension test the material fails. So the failure condition can be expressed as

$$\sigma_1 \geq \sigma_{ult}$$

Maximum shear stress theory [Guest's Theory or Tresca's Theory]

- Good for ductile materials

According to this theory when the maximum shear strength in actual case exceeds maximum allowable shear stress in simple tension test the material case. Maximum shear stress in actual case in represented as

$$\tau_{max,act} = \frac{\sigma_1 - \sigma_3}{2}$$

Maximum shear stress in simple tension case occurs at angle 45 with load, so maximum shear strength in a simple tension case can be represented as

$$\tau_{45} = \tau_{max,simp} = \frac{\sigma_y}{2}$$

Comparing these 2 quantities one can write the failure condition as

$$\frac{1}{2}(\sigma_1 - \sigma_3) \geq \frac{1}{2}\sigma_y$$

Maximum normal strain theory [St. Venant's Theory]

- Not recommended, as does not match with experimental observation.

This theory states that, when the maximum normal strain in actual case is more than maximum normal strain occurred in simple tension test case the material fails. The maximum normal strain in actual case is given by

$$strain_{max,act} = \frac{\sigma_1}{E} - v\frac{\sigma_2}{E} - v\frac{\sigma_3}{E}$$

Maximum strain in simple tension test case is given by

$$strain_{max} = \frac{\sigma_y}{E}$$

So condition of failure according to this theory is

$$\frac{\sigma_1}{E} - v\frac{\sigma_2}{E} - v\frac{\sigma_3}{E} \geq \frac{\sigma_y}{E}$$

Where *E* is Young's modulus of the material

Total strain energy theory [Haigh's Theory]

- Good for ductile material

According to this theory when the total strain energy in actual case exceeds the total strain energy in simple tension test at the time of failure, the material fails. The total strain energy in actual case is given by

$$T.S.E_{act} = \frac{1}{2E}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

The total strain energy in simple tension test at time of failure is given by

$$T.S.E_{simp} = \frac{\sigma_y{}^2}{2E}$$

So failure condition can be simplified as

$$\frac{1}{2E}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \geq \frac{\sigma_y{}^2}{2E}$$

Maximum shear strain energy theory [Mises-Henky or Von Mises' theory]

According to this theory when the shear strain energy in the actual case exceeds shear strain energy in simple tension test at the time of failure the material fails. Shear strain energy in the actual case is given by

$$S.S.E_{act} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

Shear strain energy in simple tension test at the time of failure is given by

$$S.S.E_{simp} = \frac{{\sigma_y}^2}{6G}$$

So the failure condition can be deduced as

$$\frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \geq \frac{{\sigma_y}^2}{6G}$$

Where *G* is shear modulus of the material

Out of the 5 theories discussed, the Shear strain energy theory or Von-mises theory is the most valuable one.

*Since brittle materials does not have *yield point*, you can use *ultimate tensile stress* as failure criterion.

Industrial Applications of Failure Theories

Nowadays FEA based solvers are well integrated to use failure theories. User can specify kind of failure criterion in his solution method. Shear strain energy theory is the most commonly used method. These softwares can produce Von-mises stress along material,which is based on *Shear strain energy theory*. So user can check whether maximum Von-mises stress induced in the body crosses maximum allowable stress value. It is a common practice to introduce Factor of Safety (F.S) while designing, in order to take care of worst loading scenario. So the engineer can say his design is safe if following condition satisfies.

$$von\_mises_{max} \leq \frac{\sigma_y}{F.S}$$

[Last Section adapted from http://www.learnengineering.org .- A useful site for all engineers and students]

UNIT-V

Deflection of Beams

Introduction

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

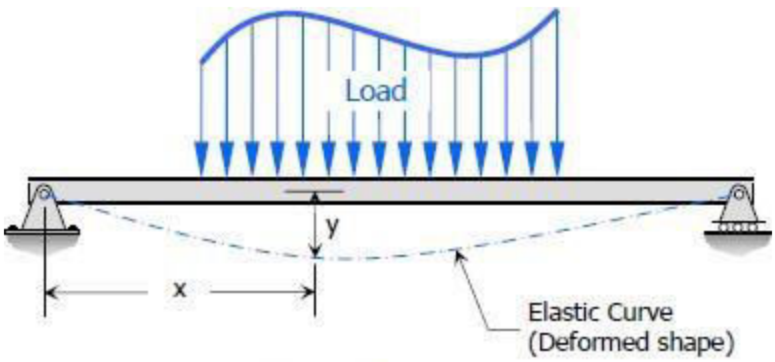


Figure: Elastic curve

A. Methods of Determining Beam Deflections

Methods for the determination of beam deflections include:

1. Double-Integration Method
2. Macaulay's Method
3. Moment-Area Method
4. Conjugate-beam Method
5. Strain - Energy method (Castigliano's Theorem)
6. Virtual work method

Of these methods, the first four shall be discussed in this course.

The stress, strain, dimension, curvature, elasticity, are all related, under certain assumption, by the theory of simple bending. This theory relates to beam flexure resulting from couples applied to the beam without consideration of the shearing forces.

B. Superposition Principle

The superposition principle is one of the most important tools for solving beam loading problems allowing simplification of very complicated design problems.

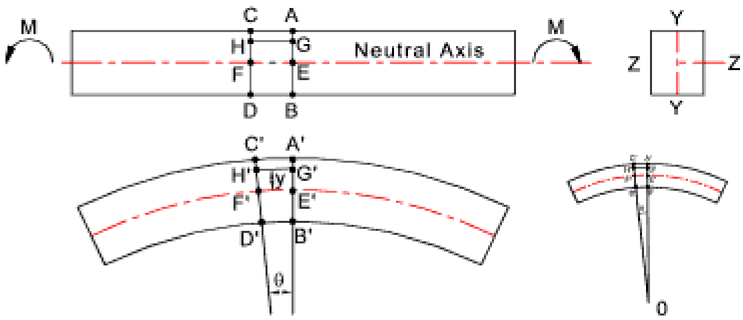
For beams subjected to several loads of different types the resulting shear force, bending moment, slope and deflection can be found at any location by summing the effects due to each load acting separately to the other loads.

C. Nomenclature

e = strain
E = Young's Modulus = σ /e (N/m^2)
y = distance of surface from neutral surface (m).
R = Radius of neutral axis (m).
I = Moment of Inertia (m^4 - more normally cm^4)
Z = section modulus = I/y_{max}(m^3 - more normally cm^3)
F = Force (N)
x = Distance along beam
δ = deflection (m)
θ = Slope
(radians) σ =
stress (N/m^2)

D. Review of Simple Bending

A straight bar of homogeneous material is subject to only a moment at one end and an equal and opposite moment at the other end...



Assumptions

The beam is symmetrical about Y-Y. The traverse plane sections remain plane and normal to the longitudinal fibres after bending (Beroulli's assumption). The fixed relationship between stress and strain (Young's Modulus)for the beam material is the same for tension and compression (σ= E.e)

Consider two section very close together (AB and CD).
After bending the sections will be at A'B' and C'D' and are no longer parallel. AC will have extended to A'C' and BD will have compressed to B'D'
The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z_Z is called the neutral axis
The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of E'F' = R
Let y be the distance(E'G') of any layer H'G' originally parallel to EF..Then

$$H'G'/E'F' =(R+y)\theta /R\ \theta = (R+y)/R$$

And the strain e at layer H'G' =

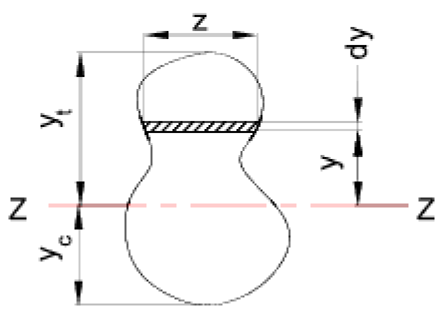
$$e = (H'G' - HG) / HG = (H'G' - HG) / EF = [(R+y)\theta - R\theta] / R\theta = y / R$$

The accepted relationship between stress and strain is $\sigma = E.e$ Therefore

$$\sigma = E.e = E.\ y$$
$$/R\ \sigma / E = y / R$$

Therefore, for the illustrated example, the tensile stress is directly related to the distance above the neutral axis. The compressive stress is also directly related to the distance below the neutral axis. Assuming E is the same for compression and tension the relationship is the same.

As the beam is in static equilibrium and is only subject to moments (no vertical shear forces) the forces across the section (AB) are entirely longitudinal and the total compressive forces must balance the total tensile forces. The internal couple resulting from the sum of ($\sigma.dA\ .y$) over the whole section must equal the externally applied moment.



$$\sum(\sigma.\delta A) = 0 \quad \text{therefore} \quad \sum(\sigma.z\ \delta y) = 0$$

$$\text{As} \quad \sigma = \frac{yE}{R} \quad \text{therefore} \quad \frac{E}{R}\sum(y.\ \delta A) = 0 \quad \text{and} \quad \frac{E}{R}\sum(y.\ z\delta y) = 0$$

This can only be correct if $\Sigma(y\delta a)$ or $\Sigma(y.z.\delta y)$ is the moment of area of the section about the neutral axis. This can only be zero if the axis passes through the centre of gravity (centroid) of the section.

The internal couple resulting from the sum of ($\sigma.dA\ .y$) over the whole section must equal the externally applied moment. Therefore the couple of the force resulting from the stress on each area when totalled over the whole area will equal the applied moment

The force on each area element = $\sigma.\ \delta A\ =\ \sigma.\ z.\ \delta y$

The resulting moment = $y.\sigma.\ \delta A\ =\ \sigma.\ z.\ y.\ \delta y$

The total moment $M = \sum(y.\sigma.\delta A)$ and $\sum(\sigma.z.y\ \delta y)$

Using $\frac{E}{R}y = \sigma$

$M = \frac{E}{R}\sum(y^2.\ \delta A)$ and $\frac{E}{R}\sum(z.y^2\delta y)$

$\sum(y^2.\delta A)$ is the Moment of Inertia of the section(I)

From the above the following important simple beam bending relationship results

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

It is clear from above that a simple beam subject to bending generates a maximum stress at the surface furthest away from the neutral axis. For sections symmetrical about Z-Z the maximum compressive and tensile stress is equal.

$$\sigma_{max} = y_{max} . M / I$$

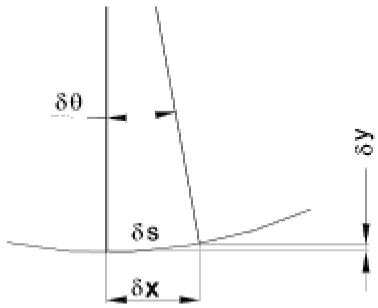
The factor I /y_{max} is given the name section Modulus (Z) and therefore

$$\sigma_{max} = M / Z$$

Values of Z are provided in the tables showing the properties of standard steel sections.

Differential Equation for the Elastic Curve

Below is shown the arc of the neutral axis of a beam subject to bending.



For small angle dy/dx = tan θ = θ
The curvature of a beam is identified as dθ /ds = 1/R
In the figure δθ is small and δx; is practically = δs; i.e ds /dx =1

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

From this simple approximation the following relationships are derived.

$$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\text{Slope} = \theta = \frac{dy}{dx} = \int\left(\frac{d^2y}{dx^2}\right)dx = \int \frac{M}{EI}\,dx$$

The deflection between limits is obtained by further integration.

$$\text{Deflection} = x = \int \theta\, dx = \int\left(\frac{dy}{dx}\right)dx = \int\int \frac{M}{EI}\,dx$$

It has been proved earlier that dM/dx = -S and dS/dx = w = - d^2M /dx^2
Where S = the shear force M is the moment and w is the distributed load /unit length of beam.
Therefore

$$S = \frac{dy}{dx}\left(EI\frac{d^2y}{dx^2}\right) = EI\frac{d^3y}{dx^3} \text{ and } -w = EI\frac{d^4y}{dx^4}$$

If w is constant or a integrable function of x then this relationship can be used to arrive at general expressions for S, M, dy/dx, or y by progressive integrations with a constant of integration being added at each stage. The properties of the supports or fixings may be used to determine the constants. (x= 0 - simply supported, dx/dy = 0 fixed end etc)

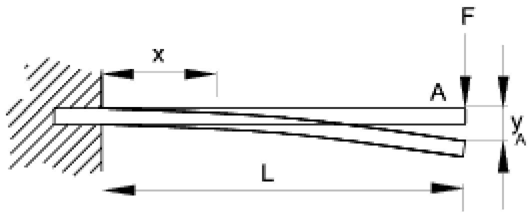
In a similar manner if an expression for the bending moment is known then the slope and deflection can be obtained at any point x by single and double integration of the relationship and applying suitable constants of integration of

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}$$

Evaluation of deflection by double-integration method

A. Example 1- Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F. At any point x along the beam there is a moment $F(x - L) = M_x = EI\, d^2y/dx^2$

$$EI\frac{d^2y}{dx^2} = -F\,(L-x) \quad \text{Integrating}$$

$$EI\frac{dy}{dx} = -F\,(Lx - \frac{x^2}{2}) + C_1 \quad .....(C_1=0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

Integrating again

$$EI\,y = -F\,(\frac{Lx^2}{2} - \frac{x^3}{6}) + C_2 \quad .....(C=0 \text{ because } y = 0 \text{ at } x = 0)$$

$$\text{At end A} \quad \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}(L^2 - \frac{L^2}{2}) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}(\frac{L^3}{2} - \frac{L^3}{6}) = -\frac{FL^3}{3EI}$$

Macaulay's Method / Singularity Functions

The basic equation governing the slope and deflection of beams is

$\frac{d^2y}{dx^2} = \frac{M}{EI}$, where M is a function of x. This is derived from the Euler-Bernoulli beam theory, based on the simplifying assumptions.

The method of integration of the above equation provides a convenient and effective way of determining the slope and deflection at any point of a beam, as long as the bending moment can be represented by a single analytical function M(x). However, when the loading of the beam is such that two different functions are needed to represent the bending moment over the entire length of the beam four constants of integration are required, and an equal number of equations, expressing continuity conditions at point of concentrated load, as well as boundary conditions at the supports A and B, must be used to determine these constants. If three or more functions were needed to represent the bending moment, additional constants and a corresponding number of additional equations would be required, resulting in rather lengthy computations. In this section these computations will be simplified through the use of the singularity functions. This is the Macaulay's method.

For general case of loadings, M(x), can be expressed in the form:

$$M(x) = M_1(x) + P_1\langle x - a_1\rangle + P_2\langle x - a_2\rangle + P_3\langle x - a_3\rangle + \dots$$

where the quantity $P_i\langle x - a_i\rangle$ represents the bending moment at the section 'x' due to point load P_i located at distance a_i from the end. The quantity $\langle x - a_i\rangle$ is a Macaulay bracket defined as

$$\langle x - a_i\rangle = \begin{cases} 0 & \text{if } x < a_i \\ x - a_i & \text{if } x > a_i \end{cases}$$

Ordinarily, when integrating $P_i(x - a_i)$ we get,

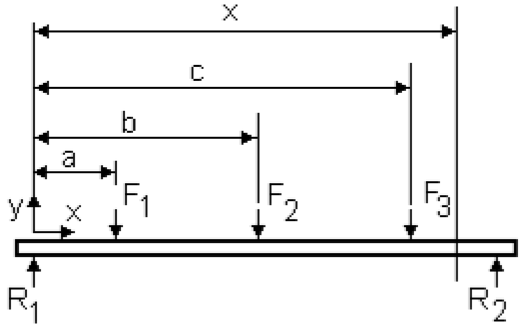
$$\int P(x-a)\ dx = P\left[\frac{x^2}{2} - ax\right] + C$$

However, when integrating expressions containing Macaulay brackets, we have to do this way:

$$\int P\langle x-a\rangle\, dx = P\frac{\langle x-a\rangle^2}{2} + C_m$$

Using these integration rules makes the calculation of the deflection of Euler-Bernoulli beams simple in situations where there are multiple point loads and point moments.

The steps for finding deflections by Macaulay's method are shown by the following example of a simply supported beam:



1. Write down the bending moment equation placing x on the extreme right hand end of the beam so that it contains all the loads. Write all terms containing x in angle brackets.

$$EI\frac{d^2y}{dx^2} = M = R_1\langle x\rangle - F_1\langle x - a\rangle - F_2\langle x - b\rangle - F_3\langle x - c\rangle$$

2. Integrate once treating the whole brackets as the variables.

$$EI\frac{dy}{dx} = R_1\frac{\langle x\rangle^2}{2} - F_1\frac{\langle x-a\rangle^2}{2} - F_2\frac{\langle x-b\rangle^2}{2} - F_3\frac{\langle x-c\rangle^2}{2} + C_1$$

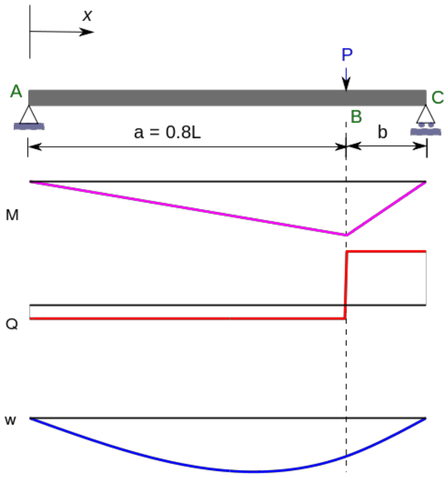
3. Integrate again using the same rules.

$$EI\,y = R_1\frac{\langle x\rangle^3}{6} - F_1\frac{\langle x-a\rangle^3}{6} - F_2\frac{\langle x-b\rangle^3}{6} - F_3\frac{\langle x-c\rangle^3}{6} + C_1x + C_2$$

4. Use boundary conditions to solve C_1 and C_2.

5. Solve slope and deflection by putting in appropriate value of x. IGNORE any brackets containing negative values.

Example 1: Simply Supported Beam with Eccentric Point Load



Consider a simply supported beam with a single eccentric concentrated load as shown in the figure.

The notations used in this example

(a) bending moment = M

(b) shear force = Q

(c) deflection = w (instead of y)

The first step is to find M. The reactions at the supports A and C are determined from the balance of forces and moments as

$$R_A + R_C = P, \quad LR_C = Pa$$

Therefore $R_A = Pb/L$ and the bending moment at a point D between A and B $(0 < x < a)$ is given by

$$M = R_A x = Pbx/L$$

Using the moment-curvature relation and the Euler-Bernoulli expression for the bending moment, we have

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L}$$

Integrating the above equation we get, for $0 < x < a$,

$$EI\frac{dw}{dx} = \frac{Pbx^2}{2L} + C_1 \qquad \text{(i)}$$

$$EIw = \frac{Pbx^3}{6L} + C_1x + C_2 \qquad \text{(ii)}$$

At $x = a_-$

$$EI\frac{dw}{dx}(a_-) = \frac{Pba^2}{2L} + C_1 \qquad \text{(iii)}$$

$$EIw(a_-) = \frac{Pba^3}{6L} + C_1a + C_2 \qquad \text{(iv)}$$

For a point D in the region BC $(a < x < L)$, the bending moment is

$$M = R_A x - P(x-a) = Pbx/L - P(x-a)$$

In Macaulay's approach we use the Macaulay bracket form of the above expression to represent the fact that a point load has been applied at location B, i.e.,

$$M = \frac{Pbx}{L} - P\langle x - a \rangle$$

Therefore the Euler-Bernoulli beam equation for this region has the form

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L} - P\langle x - a \rangle$$

Integrating the above equation, we get for $a < x < L$

$$EI\frac{dw}{dx} = \frac{Pbx^2}{2L} - P\frac{\langle x - a \rangle^2}{2} + D_1 \qquad \text{(v)}$$

$$EIw = \frac{Pbx^3}{6L} - P\frac{\langle x - a \rangle^3}{6} + D_1 x + D_2 \qquad \text{(vi)}$$

At $x = a_+$

$$EI\frac{dw}{dx}(a_+) = \frac{Pba^2}{2L} + D_1 \qquad \text{(vii)}$$

$$EIw(a_+) = \frac{Pba^3}{6L} + D_1 a + D_2 \qquad \text{(viii)}$$

Comparing equations (iii) & (vii) and (iv) & (viii) we notice that due to continuity at point B, D_1= C_1 and $D_2 = C_2$. The above observation implies that for the two regions considered, though the equation for bending moment and hence for the curvature are different, the constants of integration got during successive integration of the equation for curvature for the two regions are the same.

The above argument holds true for any number/type of discontinuities in the equations for curvature, provided that in each case the equation retains the term for the subsequent region in the form $\langle x - a \rangle^n, \langle x - b \rangle^n, \langle x - c \rangle^n$ etc. It should be remembered that for any x, giving the quantities within the brackets, as in the above case, -ve should be neglected, and the calculations should be made considering only the quantities which give +ve sign for the terms within the brackets.

Reverting to the problem, we have

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L} - P\langle x - a \rangle$$

It is obvious that the first term only is to be considered for $x < a$ and both the terms for $x > a$ and the solution is

$$EI\frac{dw}{dx} = \left[\frac{Pbx^2}{2L} + C_1\right] - \frac{P\langle x - a \rangle^2}{2}$$

$$EIw = \left[\frac{Pbx^3}{6L} + C_1 x + C_2\right] - \frac{P\langle x - a \rangle^3}{6}$$

Note that the constants are placed immediately after the first term to indicate that they go with the first term when x < a and with both the terms when x > a. The Macaulay brackets help as a reminder that the quantity on the right is zero when considering points with x < a.

Boundary conditions:

As $w = 0$ at $x = 0$, $C2 = 0$. Also, as $w = 0$ at $x = L$,

$$\left[\frac{PbL^2}{6} + C_1 L\right] - \frac{P(L-a)^3}{6} = 0$$

or,

$$C_1 = -\frac{Pb}{6L}(L^2 - b^2) \ .$$

Hence,

$$EI\frac{dw}{dx} = \left[\frac{Pbx^2}{2L} - \frac{Pb}{6L}(L^2 - b^2)\right] - \frac{P\langle x-a\rangle^2}{2}$$

$$EIw = \left[\frac{Pbx^3}{6L} - \frac{Pbx}{6L}(L^2 - b^2)\right] - \frac{P\langle x-a\rangle^3}{6}$$

Maximum Deflection:

For w to be maximum, $dw/dx = 0$. Assuming that this happens for $x < a$ we have

$$\frac{Pbx^2}{2L} - \frac{Pb}{6L}(L^2 - b^2) = 0$$

or

$$x = \pm\frac{(L^2 - b^2)^{1/2}}{\sqrt{3}}$$

Clearly $x < 0$ cannot be a solution. Therefore, the maximum deflection is given by

$$EIw_{\max} = \frac{1}{3}\left[\frac{Pb(L^2 - b^2)^{3/2}}{6\sqrt{3}L}\right] - \frac{Pb(L^2 - b^2)^{3/2}}{6\sqrt{3}L}$$

or,

$$w_{\max} = -\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL} \ .$$

Deflection at load application point

At $x = a$, i.e., at point B, the deflection is

$$EIw_B = \frac{Pba^3}{6L} - \frac{Pba}{6L}(L^2 - b^2) = \frac{Pba}{6L}(a^2 + b^2 - L^2)$$

or

$$w_B = -\frac{Pa^2b^2}{3LEI}$$

Deflection at the mid-point

It is instructive to examine the ratio of $w_{\max}/w(L/2)$. At $x = L/2$

$$EIw(L/2) = \frac{PbL^2}{48} - \frac{Pb}{12}(L^2 - b^2) = -\frac{Pb}{12}\left[\frac{3L^2}{4} - b^2\right]$$

Therefore,

$$\frac{w_{\max}}{w(L/2)} = \frac{4(L^2 - b^2)^{3/2}}{3\sqrt{3}L\left[\frac{3L^2}{4} - b^2\right]} = \frac{4(1 - \frac{b^2}{L^2})^{3/2}}{3\sqrt{3}\left[\frac{3}{4} - \frac{b^2}{L^2}\right]} = \frac{16(1 - k^2)^{3/2}}{3\sqrt{3}\,(3 - 4k^2)}$$

where k = b/L and for a < b we get 0 < k < 0.5. Even when the load is as near as 0.05L from the support, the error in estimating the deflection is only 2.6%. Hence in most of the cases the estimation of maximum deflection may be made fairly accurately with reasonable margin of error by working out deflection at the centre.

Special case of symmetrically applied load

When $a = b = L/2$, for w to be maximum

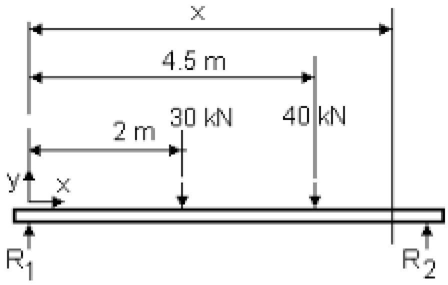
$$x = \frac{[L^2 - (L/2)^2]^{1/2}}{\sqrt{3}} = \frac{L}{2}$$

and the maximum deflection is

$$w_{\max} = -\frac{P(L/2)b[L^2 - (L/2)^2]^{3/2}}{9\sqrt{3}EIL} = -\frac{PL^3}{48EI} = w(L/2)\ .$$

Example 2: Simply Supported Beam with Two Point Loads

The beam shown is 7 m long with an EI value of 200 MN/m^2. Determine the slope and deflection at the middle of the span.



P.T.O.

SOLUTION

First solve the reactions by taking moments about the right end.

$30 \times 5 + 40 \times 2.5 = 7 R_1$ hence $R_1 = 35.71$ kN

$R_2 = 70 - 35.71 = 34.29$ kN

Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = 35710[x] - 30000[x-2] - 40000[x-4.5]$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 35710\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - 40000\frac{[x-4.5]^2}{2} + A \text{.....(1)}$$

Integrate again

$$EIy = 35710\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - 40000\frac{[x-4.5]^3}{6} + Ax + B \text{......(2)}$$

BOUNDARY CONDITIONS

$x = 0, y = 0$ and $x = 7$ $y = 0$

Using equation 2 and putting $x = 0$ and $y = 0$ we get

$$EI(0) = 35710\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - 40000\frac{[0-4.5]^3}{6} + A(0) + B$$

Ignore any bracket containing a negative value.

$0 = 0 - 0 - 0 + 0 + B$ hence $B = 0$

Using equation 2 again but this time $x=7$ and $y = 0$

$$EI(0) = 35710\frac{[7]^3}{6} - 30000\frac{[7-2]^3}{6} - 40000\frac{[7-4.5]^3}{6} + A(7) + 0$$

Evaluate A and $A = -187400$

Moment-Area Method

The moment-area theorem is a method to derive the slope, rotation and deflection of beams and frames. This theorem was developed by Mohr and later stated namely by Charles E. Greene in 1873. This method is advantageous when we solve problems involving beams, especially for those subjected to a series of concentrated loadings or having segments with different moments of inertia. If we draw the moment diagram for the beam and then divided it by the flexural rigidity(EI), the 'M/EI diagram' results by the following:

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow \theta(x) = \int \frac{M}{EI} dx$$

B. Mohr's Theorems

Theorem 1: The change in slope between any two points on the elastic curve equals the area of the $\frac{M}{EI}$ diagram between these two points.

$$\theta_{AB} = \int_A^B \frac{M}{EI} dx$$

where,

- M = bending moment expression as a function of x
- EI = flexural rigidity
- θ_{AB} = change in slope between points A and B
- A, B = points on the elastic curve

Theorem 2: The vertical deviation of a point A on an elastic curve with respect to the tangent which is extended from another point B equals the moment of the area under the M/EI diagram between those two points (A and B). This moment is computed about point A where the deviation from B to A is to be determined.

$$t_{A/B} = \int_A^B \left(\frac{M}{EI}\right) x\, dx$$

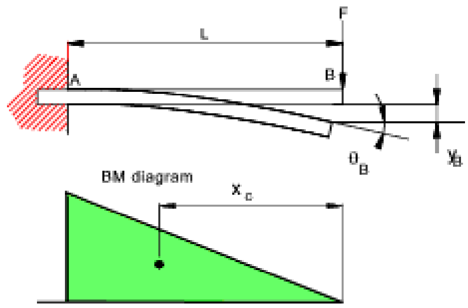
where,

- M = bending moment expression as a function of x
- EI = flexural rigidity
- $t_{A/B}$ = deviation of tangent at point B with respect to the tangent at point A

- A, B = points on the elastic curve

Two simple examples are provide below to illustrate these theorems

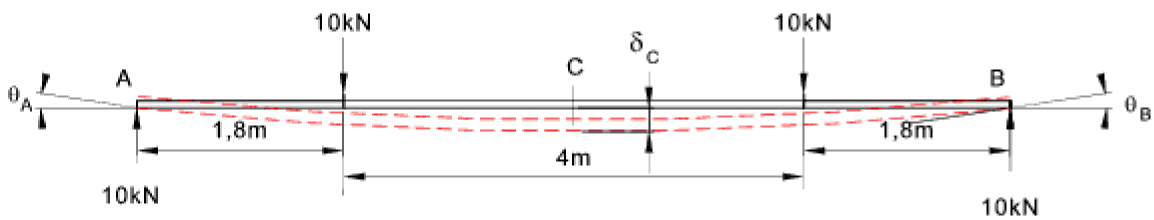
Example 1) Determine the deflection and slope of a cantilever as shown..

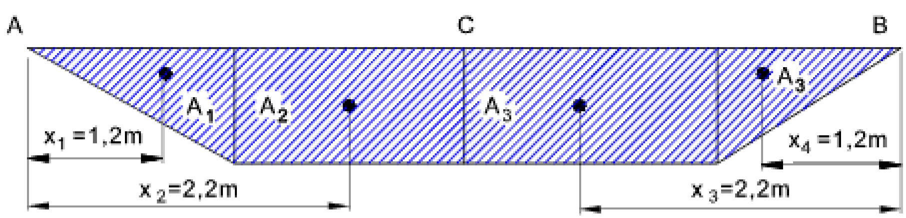


The bending moment at A = M_A = FL
The area of the bending moment diagram $A_M = F.L^2 /2$
The distance to the centroid of the BM diagram from B = x_c =
2L/3 The deflection of B = $y_b = A_M.x_c /EI = F.L^3 /3EI$
The slope at B relative to the tan at A = $\theta_b = A_M /EI = FL^2 /2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

E = 210 GPa ......I = 834 cm^4...... EI = 1,7514. $10^6 Nm^2$





Bending Moment Diagram

A_1 = 10.1, 8.1, 8/2 = 16.2kNm
A_2 = 10.1,8.2 = 36kNm
A_2 = 10.1,8.2 = 36kNm
A_1 = 10.1,8.1,8/2 = 16,2kNm

x_1 = Centroid of A_1 = (2/3).1,8 =
1,2 x_2 = Centroid of A_2 = 1,8 + 1 =
2,8 x_3 = Centroid of A_3 = 1,8 + 1 =
2,8 x_4 = Centroid of A_4 = (2/3).1,8
= 1,2

The slope at A is given by the area of the moment diagram between A and C divided by EI.

$\theta_A = (A_1 + A_2)/EI = (16{,}2+36).10^3 / (1{,}7514.\ 10^6)$
= 0,029rads = 1,7 degrees

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$\delta_C = (A_M.x_M)/EI = (A_1\, x_1 + A_2\, x_2) / EI = 120{,}24.10^3 / (1{,}7514.\ 10^6)$
= 0.0686m = 68.6mm

Conjugate Beam Method

Conjugate beam is defined as the imaginary beam with the same dimensions (length) as that of the original beam but load at any point on the conjugate beam is equal to the bending moment at that point divided by EI. The conjugate-beam method is a method to derive the slope and displacement of a beam. The conjugate-beam method was developed by H. Müller-Breslau in 1865. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope or deflection; however, this method relies only on the principles of statics, so its application will be more familiar.

We know the relationship between the load, shear and bending moment in a beam as follows:

(a) The relationship between the load 'w' at a section with the shear force 'V' at that section is

Equation 1: $\frac{dV}{dx} = -w$; and

(b) the relation between the shear force 'V' and the bending moment 'M'at that section is

$\frac{dM}{dx} = V$. Thus, by differentiating this equation we get, Equation 2: $\frac{d^2M}{dx^2} = \frac{dV}{dx} = -w$

The basis for the conjugate-beam method comes from the similarity of the above equations with the slope and deflection equations of the elastic curve.

To show this similarity, these equations are shown below.

Equation 1: $\frac{dV}{dx} = -w$	Equation 2: $\frac{d^2M}{dx^2} = -w$
Equation 3: $\frac{d\theta}{dx} = \frac{M}{EI}$	Equation 4: $\frac{d^2y}{dx^2} = \frac{M}{EI}$

Equation 1 is similar to Equation 3. And Equation 2 is similar to Equation 4. The integral forms of these equations look as follows:

Equation 1: $V = \int w\,dx$	Equation 2: $M = \int \left[\int w\,dx \right] dx$
Equation 3: $\theta = \int \left(\frac{M}{EI} \right) dx$	Equation 4: $y = \int \left[\int \left(\frac{M}{EI} \right) dx \right] dx$

Here the shear V compares with the slope θ, the moment M compares with the displacement y, and the external load w compares with the M/EI diagram.

To make use of this comparison we will now consider a beam having the same length as the real beam, but referred here as the "conjugate beam." The conjugate beam is "loaded" with the M/EI diagram derived from the load on the real beam. From the above comparisons, we can state two theorems related to the conjugate beam:

Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.

Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam

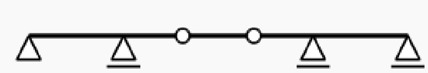
Supports of the Conjugate Beam:

When drawing the conjugate beam it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports, a consequence of Theorems 1 and 2.

Real Beam		Conjugate beam	
Fixed Support $y = 0$ $\theta = 0$		**Free End** $\overline{M} = 0$ $\overline{V} = 0$	
Free End $y \neq 0$ $\theta \neq 0$		**Fixed Support** $\overline{M} \neq 0$ $\overline{V} \neq 0$	
Hinged support $y = 0$ $\theta \neq 0$		**Hinged support** $\overline{M} = 0$ $\overline{V} \neq 0$	
Middle support $y = 0$ θ continuous		**Middle hinge** $\overline{M} = 0$ $\overline{V}$ continuous	
Middle hinge $y =$ continuous θ discontinuous		**Middle support** $\overline{M}$ continuous $\overline{V}$ discontinuous	

For example, as shown above, a pin or roller support at the end of the real beam provides zero displacement, but a non zero slope. Consequently, from Theorems 1 and 2, the conjugate beam must be supported by a pin or a roller, since this support has zero moment but has a shear or end reaction. When the real beam is fixed supported, both the slope and displacement are zero. Here the conjugate beam has a free end, since at this end there is zero shear and zero moment. Corresponding real and conjugate supports are shown below. Note that, as a rule, neglecting axial forces, statically determinate real beams have statically determinate conjugate beams; and statically indeterminate real beams have unstable conjugate beams. Although this occurs, the M/EI loading will provide the necessary "equilibrium" to hold the conjugate beam stable.

Some Examples of Conjugate Beams:

	Real beam	Conjugate beam
Simple beam		
Cantilever beam		
Left-end Overhanging beam		
Both-end overhanging beam		
Gerber's beam (2 span)		
Gerber's beam (3 span)		

Analysis Proceedure:

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the conjugate-beam method.

Conjugate beam

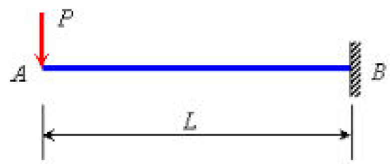
- This beam has the same length as the real beam and has corresponding supports as listed above.
- In general, if the real support allows a slope, the conjugate support must develop shear; and if the real support allows a displacement, the conjugate support must develop a moment.
- The conjugate beam is loaded with the real beam's M/EI diagram. This loading is assumed to be distributed over the conjugate beam and is directed upward when M/EI is positive and downward when M/EI is negative. In other words, the loading always acts away from the beam.

Equilibrium

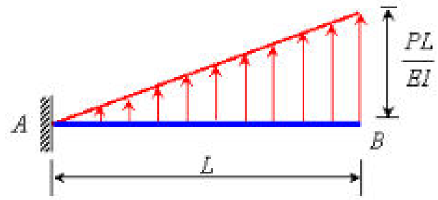
- Using the equations of statics, determine the reactions at the conjugate beams supports.
- Section the conjugate beam at the point where the slope θ and displacement Δ of the real beam are to be determined. At the section show the unknown shear V' and M' equal to θ and Δ, respectively, for the real beam. In particular, if these values are positive, and slope is counterclockwise and the displacement is upward.

Example 1:

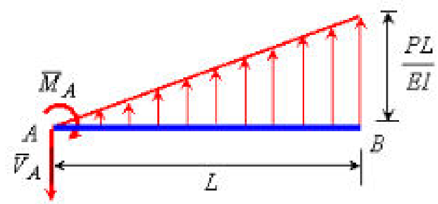
Determine the slope and deflection of point *A* of the of a cantilever beam *AB* of length *L* and uniform flexural rigidity *EI.* A concentrated force *P* is applied at the free end *of* beam.



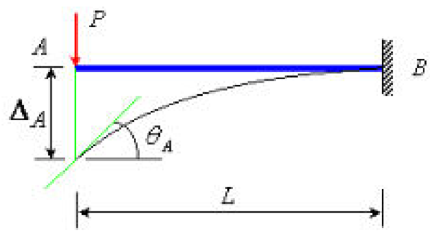
(a) A cantilever beam (actual beam)



(b) Conjugate beam (additional beam) corresponding to the actual beam



(c) Free-body diagram for the conjugate beam



(d) Deflections of the cantilever beam (actual beam)

Solution: The conjugate beam of the actual beam is shown in Figure (b). A linearly varying distributed upward elastic load with intensity equal to zero at A and equal to PL/EI at B. The free- body diagram for the conjugate beam is shown in Figure 8(c). The reactions at A of the conjugate beam are given by

$$\overline{V}_A = \frac{1}{2} \times L \times \frac{PL}{EI} = \frac{PL^2}{2EI} \left(\downarrow \right)$$

$$\overline{M}_A = \left(\frac{1}{2} \times L \times \frac{PL}{EI} \right) \times \frac{2L}{3} = \frac{PL^3}{3EI} \left[\circlearrowright \right]$$

The slope at A, and the deflection at the free end A of the actual beam in Figure (d) are respectively,

equal to the “shearing force” and the “bending moment” at the fixed end A of the conjugate beam in

Figure (c).